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On ordered groups of regular growth rates

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Abstract: We introduce an elementary class of linearly ordered groups, called growth order groups, encompassing certain groups under composition of formal series (eg transseries) as well as certain groups $\mathcal{G}_{\mathcal{M}}$ of infinitely large germs at infinity of unary functions definable in an o-minimal structure $\mathcal{M}$. We study the algebraic structure of growth order groups and give methods for constructing examples. We show that if $\mathcal{M}$ expands the real ordered field and germs in $\mathcal{G}_{\mathcal{M}}$ are levelled in the sense of Marker & Miller, then $\mathcal{G}_{\mathcal{M}}$ is a growth order group.

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Introduction

How do two quantities that grow regularly toward infinity behave under composition? How to characterise the order of growth of such magnitudes?

Hardy introduced [19] L-functions, which are real-valued functions obtained as combinations of the exponential function, the logarithm and semialgebraic functions. They naturally form a differential ring under pointwise operations. More remarkably, Hardy showed that any two such functions can always be compared on small enough neighborhoods of $+\infty$. That is, germs at $+\infty$ of L-functions are linearly ordered. For instance, the inequalities

$$\exp(t) > t^n > \cdots > t^2 > nt > \cdots > 2t > t+n > \cdots > t+1 > t$$

hold on positive half lines $(a,+\infty) \subseteq \mathbb{R}$. Differential-algebraic equations and inequalities, and indeed the whole first-order theory of fields of germs in the language of ordered valued differential fields, are well understood [3, 4].

The compositional theory of such quantities, however, is unknown. If f, g are two real-valued functions and g eventually exceeds all constant functions, then the germ of $f \circ g$ only depends on that of f and that of g. This induces a law of composition of

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germs. Even short and simple functional equations, involving germs of even regular commonplace functions... turn out to be particularly difficult to analyse. In particular, when is the simple inequality

$$f \circ g > g \circ f \tag{1}$$

satisfied for two germs f, g of L–functions? We will define a first-order theory of ordered groups of abstract regular growth rates, that describes in particular the solutions of (1) in said groups.

Let us see how the informal notion of regular growth rate can be instantiated. The most concrete example is that of elements in Hardy fields [10], ie ordered differential fields of germs. If a Hardy field $\mathscr{H}$ is closed under compositions, and if its subset $\mathscr{H}^{>\mathbb{R}}$ of germs that lie above all constant germs is closed under functional inversion, then $\mathscr{H}^{>\mathbb{R}}$ is an ordered group.

Given an o-minimal structure $\mathscr{M}$, the set $\mathscr{M}_\infty$ of germs at $+\infty$ of unary definable functions in $\mathscr{M}$ is also linearly ordered by eventual comparison. Its subset $\mathscr{G}_{\mathscr{M}}$ of germs of functions that tend to $+\infty$ at $+\infty$ is an ordered group for the induced ordering and the composition of germs, and the asymptotic growth of germs in $\mathscr{G}_{\mathscr{M}}$ is strongly related [29] to the algebra of definable sets in $\mathscr{M}$. Whereas $\mathscr{M}_\infty$ is model theoretically tame provided $\mathscr{M}$ has definable Skolem functions (see Section 3.4), the ordered group $\mathscr{G}_{\mathscr{M}}$ is not interpretable in $\mathscr{M}$ in general, and its first-order theory in the language $\mathscr{L}_{\text{og}}$ of ordered groups is not tame in general. Thirdly, consider an ordered field $\mathbb{K}$ of generalised power series [18] over an ordered field of constants C, whose set $\mathbb{K}^{>C}$ of series lying above all constants is non-empty. In certain cases, there is a composition law $\circ : \mathbb{K} \times \mathbb{K}^{>C} \longrightarrow \mathbb{K}$ such that $(\mathbb{K}^{>C}, \circ, <)$ is an ordered group. Examples include fields of transseries [21, 13], fields of hyperseries [5], and, conjecturally [6, Conclusion, 1], Conway's field of surreal numbers [11]. Groups of the form $\mathscr{H}^{>\mathbb{R}}$, $\mathscr{G}_{\mathscr{M}}$ and $\mathbb{K}^{>C}$ share important first-order properties in $\mathscr{L}_{\text{og}}$. No systematic study of this resemblance has been done yet, and this paper can be taken as a primer on that matter.

We propose a first-order theory T_{gog} in $\mathscr{L}_{\text{og}}$ whose models are ordered groups of abstract regular growth rates. We call them *growth order groups*. Simple examples include Abelian ordered groups, and, for instance, ordered groups of strictly increasing affine maps on an ordered vector space. We will show that models of T_{gog} comprise both groups of o-minimal germs, groups of formal series and more abstract examples, and that T_{gog} is sufficiently strong to provide insight on these groups that is not readily deducible from their concrete presentations.

In the first section, we give our conventions and notations for ordered groups, that are always linearly left-ordered and right-ordered. We state well-known basic facts about such groups, taking from [16, 25, 31].

In Section 2, we introduce the three axioms **GOG1**–**GOG3** for growth order groups, starting with **GOG1** and **GOG2** (Section 2.1). Section 2.2 focuses on the existence of a non-commutative valuation, in the sense of [40], on ordered groups satisfying **GOG1**. We then define *scaling elements* (Section 2.3), which form scales along which elements in the group have asymptotic expansions as in classical valuation theory. In Section 2.4, we introduce the final axiom **GOG3** and we show that growth order groups are commutative transitive [15], that is:

Theorem 1 [Theorem 2.21] *The centraliser of a non-trivial element in a growth order group is Abelian.*

We also discuss the existence of asymptotic expansions in growth order groups, and embeddings of growth order groups into groups of non-commutative formal series (Section 2.6).

Section 3 gives methods for constructing growth order groups. We give conditions under which the quotient of a growth order group is a growth order group (Section 3.2). We then define the ordered groups $\mathcal{G}_{\mathcal{M}}$ of germs in an o-minimal structure $\mathcal{M}$ and give examples where $\mathcal{G}_{\mathcal{M}}$ is, or is not a growth order group (Section 3.4).

In Section 4, we give conditions on an o-minimal expansions $\mathcal{R}$ of the real ordered field for $\mathcal{G}_{\mathcal{R}}$ to be a growth order group. Let $\mathcal{R}$ be an o-minimal expansion of the real ordered field. Given a real-valued germ g at $+\infty$ and $n \in \mathbb{N}$, we write $g^{[n]}$ for the n–fold compositional iterate of g. With [27, 39], we say that $\mathcal{R}$ is *levelled* if for all positive elements f of the ordered group $\mathcal{G}_{\mathcal{R}}$, there is an $l \in \mathbb{N}$ such that for all sufficiently large $k \in \mathbb{N}$, we have

$$-1 \leqslant \log^{[n]} \circ f - \log^{[n-l]} \leqslant 1.$$

For example, $l = 0$ for the germ of the identity or the function $0 < t \mapsto \exp(\log(t)^2)$, and $l = 1$ for the germ of exp or $\exp^2$. The main theorem is as follows:

Theorem 2 *Let $\mathcal{R}$ be an o-minimal expansion of the real ordered field. If $\mathcal{R}$ is levelled, then $\mathcal{G}_{\mathcal{R}}$ is a growth order group. Moreover, centralisers of non-trivial elements in $\mathcal{G}_{\mathcal{R}}$ are Archimedean.*

Many o-minimal expansions of $\mathbb{R}$ are levelled, including expansions of $\mathbb{R}$ by generalised analytic classes and the exponential [35, 36] (see Theorem 4.35), and the Pfaffian closure of the real ordered field [41] (see Theorem 4.33). In fact, no o-minimal expansion of $\mathbb{R}$ is known not to be levelled.

Our proof heavily relies on the fact that the elementary extension $\mathcal{R}_\infty$ of $\mathcal{R}$ is closed under derivation of germs, and that as an ordered valued differential field, it is an H-field [1]. In Section 4.1, we introduce a first-order theory of H-fields K over an ordered field of constants C with a composition law $\circ : K \times K^{>C} \longrightarrow K$ and a compositional identity $x \in K^{>C}$, such that $(K^{>C}, \circ, x, <)$ is an ordered group. A crucial feature of such fields is that they satisfy the axiom scheme of Taylor expansions (**HFC5**). We prove in Section 4.2 that certain Hardy fields closed under composition have Taylor expansions. Say that a real-valued function f is *transexponential* if the germ of f lies above $\exp^{[n]}$ for each $n \in \mathbb{N}$. We show in particular that:

Theorem 3 [Theorem 4.17] *Let $\mathcal{R}$ be an o-minimal expansion of an ordered field. Assume that $\mathcal{R}$ has an elementary substructure $\mathcal{R}_0$ with underlying ordered field $\mathbb{R}$ and that $\mathcal{R}_0$ defines no transexponential function. Then $\mathcal{R}_\infty$ has Taylor expansions.*

Using Taylor expansions, we derive conjugacy relations in H-fields with composition and inversion (Sections 4.3 and 4.4). In the case when $C = \mathbb{R}$, this allows us to prove a general result (Theorem 4.6) giving conditions under which $K^{>\mathbb{R}}$ is a growth order group. Theorem 2 follows from applications of Theorem 4.6. We rely on Miller's first dichotomy result [28] stating that either each germ in $\mathcal{R}_\infty$ is bounded by the germ of a polynomial function, or $\mathcal{R}$ defines the exponential function. The polynomially bounded and exponential cases are treated in Sections 4.5 and 4.6 respectively.

1 Ordered groups

1.1 Ordered groups

Definition 1.1 *An* **ordered group** *is a group $(\mathcal{G}, \cdot, 1)$ together with a* <u>linear</u> *(ie total) ordering $<$ on $\mathcal{G}$ such that*

$$\forall f, g, h \in \mathcal{G}, (g > h \Longrightarrow (fg > fh \wedge gf > hf)). \tag{2}$$

We write $\leqslant$ for the large relation corresponding to $<$, ie $f \leqslant g \Longleftrightarrow (f < g \vee f = g)$. Our first-order language of ordered groups is $\mathcal{L}_{\text{og}} := \langle \cdot, 1, \leqslant, \text{Inv} \rangle$ where the unary function symbol Inv is to be interpreted as the inverse map $g \mapsto g^{-1}$. We write T_{og} for the expected $\mathcal{L}_{\text{og}}$–theory of ordered groups. Homomorphisms should be understood in the model theoretic sense: a *homomorphism* of ordered groups is a nondecreasing group morphism, whereas an *embedding*, of ordered groups is a strictly increasing group morphism.

Remark 1 An ordered group $\mathcal{G}$ can be seen as a group of automorphisms of a linearly ordered set $(X,<)$ ordered by universal pointwise comparison

$$\varphi < \phi \Longleftrightarrow (\forall x \in X, (\varphi(x) < \phi(x))).$$

Indeed, let $\mathcal{G}$ act on $(\mathcal{G},<)$ by translations on the left. This intuition is particularly relevant in the case of growth order groups.

Given an ordered group $\mathcal{G}$, we write

$$\mathcal{G}^{>} := \{f \in \mathcal{G} : f > 1\} \qquad \text{and} \qquad \mathcal{G}^{\neq} := \{f \in \mathcal{G} : f \neq 1\}.$$

An ordered group $(\mathcal{G},\cdot,1,<)$ is said *Archimedean* if for all $f,g \in \mathcal{G}^{\neq}$, there is an $n \in \mathbb{Z}$ such that $f^n \geqslant g$. Recall by Hölder's theorem (see [16, Section IV.1, Theorem 1]) that $\mathcal{G}$ is Archimedean if and only if it embeds into $(\mathbb{R},+,0,<)$. In particular, Archimedean ordered groups are Abelian.

If $(H,\cdot,1)$ is a group and $f,g \in H$, then we write

$$[f,g] := f^{-1}g^{-1}fg.$$

We recall that the *centraliser* of an element $g \in H$ is the subgroup

$$\mathcal{C}(g) := \{h \in H : [g,h] = 1\} = \{h \in H : hg = gh\}.$$

For each $h \in H$, we have

$$\mathcal{C}(hgh^{-1}) = h\mathcal{C}(g)h^{-1}. \tag{3}$$

1.2 Powers

Let $(\mathcal{G},\cdot,1,<)$ be an ordered group. Let us make a few comments on powers of elements in $\mathcal{G}$. The axioms for ordered groups imply that $\mathcal{G}$ is torsion-free, ie $f^n = 1 \Longrightarrow f = 1$ for all $f \in \mathcal{G}$ and $n \in \mathbb{Z} \setminus \{0\}$.

Lemma 1.2 [31, Lemma 1.1] *For all $f,g \in \mathcal{G}$ and $n > 0$, we have*

$$[f^n,g] = 1 \Longrightarrow [f,g] = 1.$$

Corollary 1.3 [31, Corollary 1.2] *Let $f,g \in \mathcal{G}$ and $m,n \in \mathbb{N}^{>}$ with $f^n g^m = g^m f^n$. Then $fg = gf$.*

Corollary 1.4 *Let $g \in \mathscr{G}$. Let $m, n \in \mathbb{Z} \setminus \{0\}$ and $f \in \mathscr{G}$ with $f^m = g^n$. Then f is unique to satisfy $f^m = g^n$, and we have $[f, g] = 1$.*

Proof That f is unique follows from the fact that $\mathscr{G}$ is torsion-free. We have $[f, g] = 1$ by Theorem 1.3. □

2 Growth order groups

We now introduce growth order groups by defining a first-order theory $T_{\text{gog}} \supseteq T_{\text{og}}$ thereof.

2.1 Growth axioms

Let $(\mathscr{G}, \cdot, 1, <)$ be an ordered group. Consider the following sentences in $\mathscr{L}_{\text{og}}$ (after an obvious rewriting).

GOG1 Given $f, g \in \mathscr{G}^>$ with $f \geqslant g$ and $g_0 \in \mathscr{C}(g)$, there is an $f_0 \in \mathscr{C}(f)$ with $f_0 \geqslant g_0$.

GOG2 For $f, g \in \mathscr{G}^>$, we have

$$f > \mathscr{C}(g) \Longrightarrow fg > gf. \tag{4}$$

Any ordered Abelian group automatically satisfies **GOG1**, and vacuously satisfies **GOG2**. We say that $\mathscr{G}$ has *Archimedean centralisers* if for each $g \in \mathscr{G}^{\neq}$, the ordered group $\mathscr{C}(g)$ is Archimedean.

Proposition 2.1 *If $\mathscr{G}$ has Archimedean centralisers, then* **GOG1** *holds.*

Proof Let $f, g \in \mathscr{G}^>$ with $f \geqslant g$ and let $g_0 \in \mathscr{C}(g)$. We have $g^{-n} \leqslant g_0 \leqslant g^n$ for a certain $n \in \mathbb{N}$, so f^n is an element of $\mathscr{C}(f)$ with $f^n \geqslant g^n \geqslant g_0$. □

2.2 Some non-commutative valuation theory

In Sections 2.2 and 2.3, we fix an ordered group $(\mathscr{G}, \cdot, 1, <)$ satisfying **GOG1**. For $f, g \in \mathscr{G}$, we write $f \preccurlyeq g$ if $g \neq 1$ and there are $g_0, g_1 \in \mathscr{C}(g)$ such that $g_0 \leqslant f \leqslant g_1$, ie if f lies in the convex hull of $\mathscr{C}(g)$. We also set $1 \preccurlyeq g$ for all $g \in \mathscr{G}$.

Proposition 2.2 *The relation $\preccurlyeq$ is a linear quasi-ordering on $\mathcal{G}$.*

Proof Throughout the proof, we consider generic elements $f, g, h \in \mathcal{G}$.

We first prove that the relation is total. We have

$$f \preccurlyeq g \Longleftrightarrow f^{-1} \preccurlyeq g \Longleftrightarrow f \preccurlyeq g^{-1} \Longleftrightarrow f^{-1} \preccurlyeq g^{-1}.$$

Thus we may assume that $f, g > 1$. We either have $f \leqslant g$, in which case $f \preccurlyeq g$, or $g \leqslant f$, in which case $g \preccurlyeq f$.

Now suppose that $f \preccurlyeq g$ and $g \preccurlyeq h$. We may assume that $f, g, h \neq 1$. So there are $g_0, g_1 \in \mathscr{C}(g)$ and $h_0, h_1 \in \mathscr{C}(h)$ with $g_0 \leqslant f \leqslant g_1$ and $h_0 \leqslant g \leqslant h_1$. We may choose $g_0, h_0 \in \mathcal{G}^{<}$ and $g_1, h_1 \in \mathcal{G}^{>}$. By **GOG1**, there are $h_2, h_3 \in \mathscr{C}(h)$ with $g_1 \leqslant h_3$ and $g_0^{-1} \leqslant h_2$, whence $g_0 \geqslant h_2^{-1}$. We thus have $h_2^{-1} \leqslant f \leqslant h_3$, ie $f \preccurlyeq h$. So $\preccurlyeq$ is transitive. It is clearly reflexive. □

We have an equivalence relation $f \asymp g \Longleftrightarrow f \preccurlyeq g \wedge g \preccurlyeq f$ on $\mathcal{G}$ or $\mathcal{G}^{\neq}$. Given $f \in \mathcal{G}$, we write $v(f)$ for the equivalence class of f for $\asymp$, called its *valuation* and we write $v(\mathcal{G})$ for the quotient set

$$v(\mathcal{G}) = \mathcal{G}^{\neq} / \asymp = \{v(f) : f \in \mathcal{G}^{\neq}\}.$$

We write $f \prec g$ if $f \preccurlyeq g$ and $g \not\preccurlyeq f$.

Lemma 2.3 *Let $f, g \in \mathcal{G}$ with $g \neq 1$. We have $g \prec f$ if and only if $\mathscr{C}(g) < \max(f, f^{-1})$.*

Proof If $f \neq 1$, then this is immediate by definition of $\preccurlyeq$. Since $g \not\prec 1$ and $\mathscr{C}(g) \not< 1$, this yields the result. □

Proposition 2.4 *For all $g, h \in \mathcal{G}$, we have:*

(1) $g^{-1} \asymp g$.
(2) $gh \preccurlyeq g$ or $gh \preccurlyeq h$.
(3) $1 \leqslant g \leqslant h \Longrightarrow g \preccurlyeq h$.

Proof The statement 1 follows from the fact that $g^{-1} \in \mathscr{C}(g)$. Assume for contradiction that $gh \succ g$ and $gh \succ h$. We must have $g, h \neq 1$. By Lemma 2.3, we deduce that $gh > \mathscr{C}(g)$ or that $gh < \mathscr{C}(g)$. So $h > \mathscr{C}(g)$ or $h < \mathscr{C}(g)$. But then $\max(h, h^{-1})^{-2} < gh < \max(h, h^{-1})^2$. This contradicts $gh \succ h$. This shows 2. For 3 we have $h^{-1} \leqslant g \leqslant h$ where $h, h^{-1} \in \mathscr{C}(h)$. □

This shows that the function $v\colon \mathcal{G}^{\neq} \longrightarrow v(\mathcal{G})$ is a valuation in the sense of [16, Section 4.4] and of [40, Definition 2.1]. We call v the *standard valuation* on $\mathcal{G}$.

Proposition 2.5 *For $g, h \in \mathcal{G}$, we have $g \prec h \Longrightarrow gh \asymp hg \asymp h$.*

Proof We have $gh \preccurlyeq h$ by Proposition 2.4(2). Assume for contradiction that $gh \prec h$. By Proposition 2.4(1), we have $h = g^{-1}(gh) \preccurlyeq g^{-1} \asymp g \prec h$, or $h = g^{-1}(gh) \preccurlyeq gh \prec h$, which is a contradiction. Thus $gh \asymp h$. The proof of $hg \asymp h$ is symmetric. □

Proposition 2.6 *For $f \in \mathcal{G}^{\neq}$, the set $v(f) \cap \mathcal{G}^{>}$ is convex.*

Proof Let $g, h \in \mathcal{G}^{>}$ with $g, h \asymp f$ and let $j \in \mathcal{G}$ with $g \leqslant j \leqslant h$. We have $g \preccurlyeq j$ and $j \preccurlyeq h$ by Theorem 2.4(3). So $f \preccurlyeq j$ and $j \preccurlyeq f$ by Theorem 2.2, whence $j \asymp f$. □

We can thus define a linear ordering $<$ on $v(\mathcal{G})$, where for $g, h \in \mathcal{G}^{\neq}$, we set $v(g) < v(h)$ if and only if $g \prec h$, ie if $v(g) \cap \mathcal{G}^{>} < v(h) \cap \mathcal{G}^{>}$.

Definition 2.7 *The* **value set** *of $\mathcal{G}$ is the (order type of the) linearly ordered set* $(v(\mathcal{G}), <)$.

One sees that $\mathcal{G}$ has value set 0 if and only if it is trivial, and that non-trivial Abelian ordered groups have value set 1.

Example 2.8 Let $\mathscr{R}$ denote the real ordered field. It will follow from Theorem 2 that $\mathcal{G}_{\mathscr{R}}$ satisfies **GOG1** and has Archimedean centralisers. Therefore, the convex hull of $\mathcal{C}(g)$ for $g \in \mathcal{G}_{\mathscr{R}}$ is simply the convex hull of the set $g^{[\mathbb{Z}]}$ of iterates of g and its inverse. Definable functions in $\mathscr{R}$ are semialgebraic. Any non-trivial semialgeraic function f satisfies $\lim_{t\to+\infty} \frac{f(t)}{rt^q} = 1$ for an $(r, q) \in \mathbb{R}^{\times} \times \mathbb{Q}$ (as it has a Puiseux series expansion). Therefore the valuation of the square function is maximal in $c(\mathcal{G})$ and $v(2\,\mathrm{id})$ is maximal in $v(\mathcal{G}_{\mathscr{R}}) \setminus \{v(\mathrm{id}^2)\}$. Applying the same idea to $f - \mathrm{id}$ for $f \in \mathcal{G}$, we see that

$$\begin{array}{rcl} \{q \in \mathbb{Q} : q < 1\} & \longrightarrow & v(\mathcal{G}) \setminus \{v(\mathrm{id}^2), v(2\,\mathrm{id})\} \\ q & \longmapsto & v(\mathrm{id} + \mathrm{id}^q) \end{array}$$

is an isomorphism of ordered sets. In other words, the value set of $\mathcal{G}_{\mathscr{R}}$ is the rational interval $((-\infty, 1] \cup \{2\}, <)$.

Lemma 2.9 *For $f, g \in \mathcal{G}$ with $g \prec f$, we have $fgf^{-1} \prec f$.*

Proof The conjugation by f is an automorphism of $\mathscr{G}$ and $\prec$ is $\varnothing$–definable in the language of ordered groups. $\square$

Given $g, h \in \mathscr{G}^{\neq}$, we write

$$g \sim h \qquad \text{if and only if} \qquad gh^{-1} \prec g.$$

Lemma 2.10 *For all $g, h \in \mathscr{G}^{\neq}$, the following are equivalent:*

(1) $g \sim h$
(2) $gh^{-1} \prec h$
(3) $hg^{-1} \prec g$
(4) $h \sim g$.

Proof Suppose that $g \sim h$, ie $gh^{-1} \prec g$. We cannot have $h^{-1} \prec g$ by Theorem 2.5, so we also have $gh^{-1} \prec h^{-1} \asymp h$ by Theorem 2.2. We deduce that 1 and 2 are equivalent. Likewise 3 and 4 are equivalent. Since $gh^{-1} \asymp hg^{-1}$ by Proposition 2.4(1), the statements 1 and 3 are equivalent. This concludes the proof. $\square$

Corollary 2.11 *For $g, h \in \mathscr{G}^{\neq}$, we have $g \sim h \Longleftrightarrow g^{-1} \sim h^{-1}$.*

Note that for $g, h \in \mathscr{G}$ with $g \sim h$, we have $g \asymp h$. We have $gh^{-1} \prec h$ by Theorem 2.10, so $g = (gh^{-1})h \asymp h$ by Theorem 2.5.

Lemma 2.12 *The relation $\sim$ is an equivalence relation on $\mathscr{G}^{\neq}$.*

Proof For all $g \in \mathscr{G}^{\neq}$, we have $1 \prec g$ whence $g \sim g$. Theorem 2.10 implies that $\sim$ is symmetric. Let $f, g, h \in \mathscr{G}^{\neq}$ with $f \sim g$ and $g \sim h$. So $f \asymp g \asymp h$. We have $fh^{-1} = (fg^{-1})(gh^{-1})$ where $(fg^{-1}), (gh^{-1}) \prec f$ so $fh^{-1} \prec f$ by Theorem 2.4(2). So $f \sim h$, ie $\sim$ is transitive. $\square$

Given a $g \in \mathscr{G}^{\neq}$, we write $\operatorname{res}(g)$ for the equivalence class of g for $\sim$ in $\mathscr{G}^{\neq}$. We call $\operatorname{res}(g)$ the *residue* of g.

Proposition 2.13 *For $g \in \mathscr{G}^{\neq}$, the set $\operatorname{res}(g)$ is convex.*

Proof Let $f, h \in \mathscr{G}^{\neq}$ with $f \sim g \sim h$ and $j \in \mathscr{G}^{\neq}$ with $f \leqslant j \leqslant h$. In view of Theorem 2.11, we may assume that $g > 1$. Consider an $s_0 \in \mathscr{C}(jg^{-1})$. Suppose that $jg^{-1} \geqslant 1$. Since $hg^{-1} \geqslant jg^{-1}$, we find by **GOG1** an $h_0 \in \mathscr{C}(hg^{-1})$ with $h_0 \geqslant s_0$. Now $hg^{-1} \prec g$ so $h_0 < g$, so $s_0 < g$. This shows that $jg^{-1} \prec g$, whence $j \sim g$ in that case. Suppose now that $jg^{-1} \leqslant 1$. So $1 \leqslant gj^{-1} \leqslant gf^{-1}$. But $gf^{-1} \prec g$ so the same arguments for gj^{-1} show that $s_0 < g$, whence $j \sim g$. So $\operatorname{res}(g)$ is convex. $\square$

We can thus define a linear ordering $\lessdot$ on $\mathrm{res}(\mathcal{G}) := \mathcal{G}^{\neq} / \sim$ given by

$$\mathrm{res}(f) \lessdot \mathrm{res}(g) \Longleftrightarrow f < g \wedge f \nsim g.$$

We set $\mathrm{res}(1) = \{1\}$ and $\{1\} \lessdot \mathrm{res}(f)$ for all $f \in \mathcal{G}^{\neq}$. We also write $f \lessdot g$ whenever $\mathrm{res}(f) \lessdot \mathrm{res}(g)$. Although we will not rely on this fact, this is also strict ordering on $\mathcal{G}$

Lemma 2.14 *Let $g, h \in \mathcal{G}^{\neq}$ with $g \sim h^{-1}$ or $g \prec h$. Then $[g, h] \prec h$.*

Proof First suppose that $g \sim h^{-1}$. Theorem 2.10 gives $g^{-1}h^{-1}, gh \prec h$. So $[g, h] \prec h$ by Proposition 2.4(2). Suppose now that $g \prec h$. So $\delta := h^{-1}gh \prec h$. We obtain

$$[g, h] = g^{-1}h^{-1}gh = g^{-1}\delta \prec h$$

by Proposition 2.4(2). □

2.3 Scaling elements

Recall that $\mathcal{G}$ is an ordered group satisfying **GOG1**.

Definition 2.15 *We say that an element $\mathfrak{s} \in \mathcal{G}^{>}$ is* ***scaling*** *if $\mathscr{C}(\mathfrak{s})$ is Abelian, and for all $f \in \mathcal{G}$ with $f \asymp \mathfrak{s}$, there is a $g \in \mathscr{C}(\mathfrak{s})^{\neq}$ with $g \sim f$.*

Given a scaling $\mathfrak{s}$ and $f \asymp g$, the element g is unique in $\mathscr{C}(\mathfrak{s})$. Indeed, for $h \in \mathscr{C}(\mathfrak{s}) \setminus \{g\}$, writing $j := hg^{-1}$ we have $f(jg)^{-1} \asymp h \asymp f$ by Theorem 2.4(1, 2), so we do not have $h = jg \sim f$. Note that each positive element in an Abelian ordered group is scaling.

Definition 2.16 *We say that $\mathcal{G}$* **has scaling elements** *if for all $\rho \in v(\mathcal{G})$, there is an $\mathfrak{s} \in \rho$ which is scaling.*

Proposition 2.17 *Let $\mathfrak{s} \in \mathcal{G}^{>}$ such that $(\mathscr{C}(\mathfrak{s}), \cdot, 1, <)$ is isomorphic to $(\mathbb{R}, +, 0, <)$. Then $\mathfrak{s}$ is scaling.*

Proof Let $f \in \mathcal{G}^{\neq}$ with $f \asymp \mathfrak{s}$. If $f \in \mathscr{C}(\mathfrak{s})$, then we are done. Assume that $f \notin \mathscr{C}(\mathfrak{s})$ and set $h := \sup\{g \in \mathscr{C}(\mathfrak{s}) : g \leqslant f\}$. For $\varepsilon \in \mathscr{C}(\mathfrak{s}) \cap \mathcal{G}^{>}$, we have

$$\varepsilon^{-1}h, h\varepsilon^{-1} < h < \varepsilon h, h\varepsilon,$$

because $(\mathcal{G}, \cdot, 1, <)$ is an ordered group. We deduce that

$$\mathscr{C}(\mathfrak{s}) \cap \mathcal{G}^{<} < h^{-1}f < \mathscr{C}(\mathfrak{s}) \cap \mathcal{G}^{>}. \tag{5}$$

Assume for contradiction that $h^{-1}f \succcurlyeq f$. We have $h \in \mathscr{C}(\mathfrak{s})$ so $h \asymp \mathfrak{s} \asymp f$. By Theorem 2.4(1, 2) we have $h^{-1}f \asymp f$. Since $h \in \mathscr{C}(\mathfrak{s})$ and $f \notin \mathscr{C}(\mathfrak{s})$, there are f_0, f_1 which have the same sign, with $f_0 < h^{-1}f < f_1$. By **GOG1**, there are $g_0, g_1 \in \mathscr{C}(\mathfrak{s})$ which have the same sign as well, with $g_0 < h^{-1}f < g_1$. This contradicts (5). We deduce that $h^{-1}f \prec f$, ie $h \sim f$. □

Lemma 2.18 *Suppose that $\mathfrak{s} \in \mathscr{G}^{>}$ is scaling. Then for all $f, g \in \mathscr{G}^{\neq}$ with $f \asymp g \asymp \mathfrak{s}$, we have $[f, g] \prec f$.*

Proof If $f \sim g^{-1}$, then this follows from Theorem 2.14. Assume that $f \nsim g^{-1}$. Let $t, u \in \mathscr{C}(\mathfrak{s})^{\neq}$ with $t \sim f$ and $u \sim g$. We have $t \nsim u^{-1}$ by Theorem 2.13, so Proposition 2.4(2) implies that $tu \asymp t \asymp \mathfrak{s}$. Set

$$\begin{aligned} \varepsilon &:= t^{-1}f \prec \mathfrak{s} \\ \delta &:= gu^{-1} \prec \mathfrak{s}. \end{aligned}$$

Recall that $\mathscr{C}(\mathfrak{s})$ is Abelian, so $[t, u] = 1$. We have

$$\begin{aligned} [f, g] &= f^{-1}g^{-1}fg \\ &= \varepsilon^{-1}t^{-1}u^{-1}\delta^{-1}t\varepsilon\delta u \\ &= \varepsilon^{-1}[t, u](u^{-1}(t^{-1}\delta^{-1}t)\varepsilon\delta u) \\ &= \varepsilon^{-1}(u^{-1}(t^{-1}\delta^{-1}t\varepsilon\delta)u). \end{aligned}$$

Now $\delta \prec t$ so $t^{-1}\delta^{-1}t \prec t$ by Lemma 2.9, so $t^{-1}\delta^{-1}t\varepsilon\delta \prec t$ by Proposition 2.4(2), so $u^{-1}(t^{-1}\delta^{-1}t\varepsilon\delta)u \prec t$ by Lemma 2.9, whence finally $[f, g] \prec t \asymp f$ by Proposition 2.4(2). □

Proposition 2.19 *If $\mathfrak{s} \in \mathscr{G}^{>}$ is scaling, then the centraliser of each $f \asymp \mathfrak{s}$ is commutative.*

Proof Let $f \asymp \mathfrak{s}$ and let $g, h \in \mathscr{C}(f)$. Assume for contradiction that $[g, h] \neq 1$. Then, since $[g, h] \in \mathscr{C}(f)$, we have $[g, h] \asymp f$. This contradicts Theorem 2.18. □

2.4 Growth order groups

Given an ordered group $(\mathscr{G}, \cdot, 1, <)$, we consider the following axiomatic property:

GOG3 $\mathscr{G}$ has scaling elements.

Using the $\mathcal{L}_{\mathrm{og}}$–definable abbreviations $\sim$ and $\asymp$, a natural first-order formulation of **GOG3** is $\forall a \exists b \forall c \exists d (a \neq 1 \to ((c \asymp a) \to ([d,b] = 1 \wedge c \sim d)))$.

Definition 2.20 *We say that an ordered group* $(\mathcal{G}, \cdot, 1, <)$ *is a* **growth order group** *if it satisfies* **GOG1**, **GOG2** *and* **GOG3**.

All Abelian ordered groups are growth order groups. We write T_{gog} for the $\mathcal{L}_{\mathrm{og}}$–theory $T_{\mathrm{og}} \cup \{$**GOG1**, **GOG2**, **GOG3**$\}$. A CT-group is a group in which centralisers of non-trivial elements are Abelian. As corollaries of Theorem 2.19, we have:

Corollary 2.21 *Growth order groups are CT-groups.*

Corollary 2.22 *Any non-Abelian growth order group has trivial center.*

2.5 Skeletons

Let $\mathcal{G}$ be a growth order group. Fix a $\rho \in v(\mathcal{G})$ and consider the set

$$\mathscr{C}_\rho := \{\mathrm{res}(f) : v(f) = \rho \vee f = 1\}.$$

Recall that $(\mathscr{C}_\rho, \lessdot)$ is linearly ordered. For all $\mathrm{res}(f), \mathrm{res}(g) \in \mathscr{C}_\rho$, we set

$$\mathrm{res}(f) + \mathrm{res}(g) := \mathrm{res}(fg)$$

if $fg \asymp f$ and $\mathrm{res}(f) + \mathrm{res}(g) := \mathrm{res}(1)$ if $fg \prec f$.

Lemma 2.23 *The structure* $(\mathscr{C}_\rho, +, \mathrm{res}(1), \lessdot)$ *is an ordered Abelian group. Moreover, given a scaling element* $\mathfrak{s}$ *in* $\mathcal{G}$, *the function* $\varphi_{\mathfrak{s}} : \mathscr{C}(\mathfrak{s}) \longrightarrow \mathscr{C}_{v(\mathfrak{s})}\ ; f \mapsto \mathrm{res}(f)$ *is an isomorphism.*

Proof The operation $+ : \mathscr{C}_\rho \times \mathscr{C}_\rho \longrightarrow \mathscr{C}_\rho$ is well-defined. For $\mathrm{res}(h) \in \mathscr{C}_\rho$ where $v(h) = \rho$, since $\mathfrak{s}$ is scaling, there is a unique $f \in \mathscr{C}(\mathfrak{s})$ with $h \sim f$, whence $\mathrm{res}(f) = \mathrm{res}(h)$. So $\varphi_{\mathfrak{s}}$ is surjective. Let $f, g \in \mathscr{C}(\mathfrak{s})$. Note that $fg \in \mathscr{C}(\mathfrak{s})$, so $fg = 1$ or $fg \asymp \mathfrak{s}$. We thus have $fg = 1 \Longleftrightarrow fg \prec f \Longleftrightarrow v(fg) < \rho$. So $\mathrm{res}(fg) = \mathrm{res}(f) + \mathrm{res}(g)$. If $1 < f$, then $1 \lessdot f$, so $\mathrm{res}(1) \lessdot \mathrm{res}(f)$. Altogether this shows that $\varphi_{\mathfrak{s}}$ is an isomorphism between the $\mathcal{L}_{\mathrm{og}}$–structures $(\mathscr{C}(\mathfrak{s}), \cdot, 1, <)$ and $(\mathscr{C}_\rho, +, \mathrm{res}(1), \lessdot)$. In particular, the latter is a an ordered Abelian group. $\square$

We call $(\mathscr{C}_\rho)_{\rho \in v(\mathscr{G})}$ the *skeleton* of $\mathscr{G}$. If $\mathscr{H}$ is a growth order group, then each ordered group homomorphism $\Phi : \mathscr{G} \longrightarrow \mathscr{H}$ induces a homomorphism of skeletons, ie a nondecreasing map

$$\Phi_v : v(\mathscr{G}) \longrightarrow v(\mathscr{H})\,;\ v(g) \longmapsto v(\Phi(g))$$

and, for each $\rho \in v(\mathscr{G})$, an ordered group homomorphism

$$\Phi_\rho : \mathscr{C}_\rho \longrightarrow \mathscr{C}_{\Phi_v(\rho)} \quad ;\ \mathrm{res}(f) \mapsto \mathrm{res}(\Phi(f)).$$

2.6 On the structure of growth order groups

Any ordered group is [22, Theorem 1] a quotient by a convex normal subgroup of an ordered free group. However that description is far from being as precise and concrete as the Hahn embedding theorem [18] (see [17, Theorem 4.C]) for Abelian ordered groups, which construes them as lexicographically ordered groups of formal commutative series with real coefficients. We expect that a similar description exists for growth order groups, as we next explain.

Let $\mathscr{G}$ be a growth order group, and let $\mathcal{S}$ be a set of unique scaling representatives for each valuation. Given $\mathfrak{s} \in \mathcal{S}$ and $c = \mathrm{res}(g) \in \mathscr{C}^{\neq}_{v(\mathfrak{s})}$, we let $\mathfrak{s}^{[c]}$ denote the unique element of $\mathscr{C}(\mathfrak{s})$ with $\mathfrak{s}^{[c]} \sim g$. We also write $\mathfrak{s}^{[0]} := 1$. Given $f_0 \in \mathscr{G}^{\neq}$, there are a unique $\mathfrak{s}_0 \in \mathcal{S}$ with $\mathfrak{s}_0 \asymp f_0$ and a unique $c_0 \in \mathscr{C}_{v(\mathfrak{s}_0)}$ with $f_0 \sim \mathfrak{s}_0^{[c_0]}$. Define

$$f_1 := \mathfrak{s}^{[-c_0]} f_0. \tag{6}$$

Reiterating the process for f_1 if $f_1 \neq 1$ and continuing further, we obtain an $\ell \leqslant \omega$, a strictly $\prec$–decreasing sequence $(\mathfrak{s}_n)_{n<\ell}$ in $\mathcal{S}$ and a sequence $(c_n)_{n<\ell} \in \prod_{n<\ell} \mathscr{C}_{v(\mathfrak{s}_n)}$ with

$$f_0 \approx \mathfrak{s}_0^{[c_0]} \mathfrak{s}_1^{[c_1]} \mathfrak{s}_2^{[c_2]} \cdots \mathfrak{s}_n^{[c_n]} \cdots ,$$

in the sense that $(\mathfrak{s}_0^{[c_0]} \mathfrak{s}_1^{[c_1]} \mathfrak{s}_2^{[c_2]} \cdots \mathfrak{s}_n^{[c_n]})^{-1} f_0 \prec \mathfrak{s}_n$ whenever $n < \ell$. If $\ell = \omega$, then there may exist several elements of $\mathscr{G}^{\neq}$ with the same expansion as f_0 (consider for instance an ultrapower of $\mathscr{G}$), so describing f_0 in full entails extending this process inductively. This points to the existence of an embedding of $\mathscr{G}$ into an ordered group of formal non-commutative series

$$\mathfrak{g}_0^{[c_0]} \mathfrak{g}_1^{[c_1]} \cdots \mathfrak{g}_\gamma^{[c_\gamma]} \cdots ,\ \gamma < \lambda \tag{7}$$

where $(c_\gamma)_{\gamma<\lambda} \in \prod_{\gamma<\lambda} \mathscr{C}_{\mathfrak{g}_\gamma}$, $(\mathfrak{g}_\gamma)_{\gamma<\lambda} \in v(\mathscr{G})^\lambda$ is strictly decreasing and λ is an ordinal. In other words, it is conceivable that there is a non-commutative version of the Hahn embedding theorem for growth order groups.

The construction of such an ordered group is difficult, and it requires additional information besides the skeleton. Moreover, several issues that are absent in the Abelian case appear here.

First, the choice in (6) of expanding f_0 systematically on the right is arbitrary. One could expand f_0 on the left, or even alternate choices. Indeed, given an infinite limit ordinal κ and a function $N : \kappa \longrightarrow \{\text{left}, \text{right}\}$, one may expand f_0 on the side prescribed by $N(\gamma)$ at each stage $\gamma < \kappa$. This induces a linear ordering on κ which we call tree-like. How can one describe series with tree-like support?

Secondly, studying examples of groups of transseries shows that in certain cases, extending $\mathcal{G}$ with transfinite expansions as in (7) forces the existence of valuations that are not comparable to elements in $\mathcal{G}$. More precisely, there can be series $s := \mathfrak{g}_0^{[c_0]}\mathfrak{g}_1^{[c_1]} \cdots \mathfrak{g}_\gamma^{[c_\gamma]} \cdots$ and elements $\mathfrak{g} \in v(\mathcal{G})$ such that the valuation of $s\mathfrak{g}^{[c]}s^{-1}$ should lie in an unfilled cut in $(v(\mathcal{G}), \prec)$. So an embedding theorem must involve constraints on the skeleton of $\mathcal{G}$.

Question 1 Embedding problem. For a linearly ordered set $(I, <)$ and a family $(\mathscr{C}_i)_{i\in I}$ of Abelian ordered groups, under what conditions can one define a group law $*$ on the set $\mathbf{H}_{i\in I}\mathscr{C}_i$ of functions $f \in \Pi_{i\in I}\mathscr{C}_i$ with anti-well-ordered support $\operatorname{supp} f = \{i \in I : f(i) \in \mathscr{C}_i \setminus \{0\}\}$, ordered lexicographically, such that

- $(\mathbf{H}_{i\in I}\mathscr{C}_i, *, 1, <)$ is a growth order group with skeleton $\simeq (\mathscr{C}_i)_{i\in I}$,
- for all growth order groups $\mathcal{G}$ with skeleton $\simeq (\mathscr{C}_i)_{i\in I}$, there is an embedding of ordered groups $\mathcal{G} \longrightarrow \mathbf{H}_{i\in I}\mathscr{C}_i$?

As a first step toward answering this question, we showed [7] that certain groups of transseries can be represented as groups $(\mathbf{H}_{i\in I}\mathscr{C}_i, *, 1, <)$.

3 Constructions of growth ordered groups

We now give methods for constructing growth order groups.

Example 3.1 We constructed [5] an ordered field of formal series $\tilde{\mathbb{L}}$ equipped with a composition law $\circ : \tilde{\mathbb{L}} \times \tilde{\mathbb{L}}^{>\mathbb{R}} \longrightarrow \tilde{\mathbb{L}}$ and showed [5, Propositions 9.23 and 10.25] that $(\tilde{\mathbb{L}}^{>\mathbb{R}}, \circ, <)$ is a growth order group with Archimedean centralisers [5, Proposition 10.24].

3.1 Semidirect products

Let $(\mathcal{G}, \cdot, 1, <)$, $(G, +, 0, <)$ be ordered groups. For clarity, we will use additive denotation for G, but we do not assume that $(G, +, 0)$ is Abelian. Let a morphism $\rho : (\mathcal{G}, \cdot, 1) \longrightarrow \mathrm{Aut}(G, +, 0)$ be given with the following properties:

MGA1 Each $\rho(g), g \in \mathcal{G}$ is strictly increasing.

MGA2 For $f, g \in \mathcal{G}$ with $f < g$ and $a \in G^{>}$ we have $\rho(f)(a) < \rho(g)(a)$.

For $(g, a) \in \mathcal{G} \times G$, we write

$$g * a := \rho(g)(a).$$

We consider the lexicographically ordered semidirect product $\mathcal{G} \ltimes_\rho G$, ie the Cartesian product $\mathcal{G} \times G$ equipped with the operation

$$\forall (f, a), (g, b) \in \mathcal{G} \times G, (f, a) \cdot (g, b) := (fg, (f * a) + b),$$

and the lexicographic ordering

$$\forall (g, a), (h, b) \in \mathcal{G} \times G, (g, a) < (h, b) \Longleftrightarrow (g < h \text{ or } (g = h \text{ and } a < b)).$$

Note that the inverse of an $(f, a) \in \mathcal{G} \ltimes_\rho G$ is given by

$$(f, a)^{-1} = (f^{-1}, f^{-1} * (-a)).$$

Proposition 3.2 *The structure* $(\mathcal{G} \ltimes_\rho G, \cdot, (1, 0), <)$ *is an ordered group, and the functions*

$$G \longrightarrow \mathcal{G} \ltimes_\rho G; a \mapsto (1, a) \qquad \text{and} \qquad \mathcal{G} \longrightarrow \mathcal{G} \ltimes_\rho G; f \mapsto (f, 0)$$

are embeddings.

Proof The lexicographic ordering is linear, so we need only show that $\mathcal{G} \ltimes_\rho G$ is an ordered group. Assume that $(g, b) > (h, c)$. If $g > h$, then $fg > fh$ and $gf > hf$, so $(f, a) \cdot (g, b) > (f, a) \cdot (h, c)$ and $(g, b) \cdot (f, a) > (h, c) \cdot (f, a)$. Otherwise $g = h$ and $b > c$, so $f * b > f * c$, whence

$$(f, a) \cdot (g, b) = (fh, f * b + a) > (fh, f * c + a) = (f, a) \cdot (h, c).$$

Likewise $f * a + b > f * a + c$ so

$$(g, b) \cdot (f, a) = (hf, f * a + b) > (hf, f * a + c) = (h, c) \cdot (f, a).$$

This shows that $\mathcal{G} \ltimes_\rho G$ is an ordered group. It is easily checked that the two functions above are embeddings. □

We consider two further conditions on $(\mathscr{G}, G)$:

MGA3 For all $g \in \mathscr{G}$ and $b \in G$, for sufficiently large $g' \in \mathscr{C}(g)$, there is a $b' \in G$ with

$$g * b' + b = g' * b + b'.$$

MGA4 For all $a \in G, f \in \mathscr{G}^{>}$ and $b \in G^{>}$, we have

$$f * b > a + b - a.$$

Remark 2 Let $(g, b) \in (\mathscr{G} \rtimes_\rho G)^{\neq}$. For $(f, a) \in \mathscr{G} \rtimes_\rho G$, we have $(f, a) \cdot (g, b) = (g, b) \cdot (f, a)$ if and only if

$$fg = gf \qquad \text{and} \qquad f * b + a = g * a + b.$$

The first condition means that $f \in \mathscr{C}(g)$. Now given $h \in \mathscr{C}(g)^{>}$ sufficiently large, by **MGA3**, there is an $a \in G$ with $g * a - a = h * b - b$, hence $(h, a) \in \mathscr{C}(g, b)$.

Remark 3 If G is Abelian, then **MGA4** follows from **MGA2**.

Proposition 3.3 *Let ρ satisfy* **MGA1**–**MGA4**. *If $\mathscr{G}$ and G are growth order groups, then so is $(\mathscr{G} \rtimes_\rho G, \cdot, (1,0), <)$.*

Proof We first prove **GOG1**. Let $(f, a), (g, b) \in \mathscr{G}^{>}$ with $(f, a) > (g, b)$, and let $(g', b') \in \mathscr{C}(g, b)$. Assume first that $g = 1$, so $b > 1$. Assume for contradiction that $g' > 1$. then we have $(g', g' * b + b') = (g', b') \cdot (1, b) = (1, b) \cdot (g', b') = (g', b' + b)$, so $g' * b = b' + b - b'$. But this contradicts **MGA4**. If $g' < 1$, then $(f, a) > (g', b')$. So we may assume that $g' = 1$. If $f > 1$, then $(f, a) > (g', b')$. Otherwise, we must have $f = 1$ and thus $a > b$. Now **GOG1** in G gives an $a' \in \mathscr{C}(a)$ with $a' > b'$, whence $(1, a') \in \mathscr{C}(1, a)$ and $(1, a') > (1, b')$. We now treat the case when $g > 1$. We have $g' \in \mathscr{C}(g)$ where $f \geqslant g$, so by **GOG1** in $\mathscr{G}$ there is an $f' \in \mathscr{C}(f)$ with $f' \geqslant g'$. In view of Remark 2, we may choose f' sufficiently large so that $f' > g'$ and that there be an $a' \in G$ with $(f', a') \in \mathscr{C}(f, a)$. We have $(f', a') > (g', b')$, hence **GOG1** holds in $\mathscr{G} \rtimes_\rho G$.

Let $(g, b), (f, a) \in \mathscr{G} \rtimes_\rho G$ with $(f, a) > \mathscr{C}(g, b)$ and $(g, b) > (1, 0)$. Assume that $g = 1$, so $b > 0$ and $f \geqslant 1$. We have

$$\begin{aligned}
(f, a) \cdot (1, b) \cdot (f, a)^{-1} &= (f, a) \cdot (1, b) \cdot (f^{-1}, f^{-1} * (-a)) \\
&= (1, f * (f^{-1} * (-a) + b) + a) \\
&= (1, (-a) + f * b + a).
\end{aligned}$$

If $f=1$, then the condition $(1,a)>\mathscr{C}(1,b)$ amounts to $a>\mathscr{C}(b)$, so **GOG2** in G gives $(-a)+b+a>b$. That is,

$$(f,a)\cdot(1,b)\cdot(f,a)^{-1}>(1,b).$$

If $f>1$, then **MGA4** gives $(-a)+f*b+a>b$, whence again

$$(f,a)\cdot(1,b)\cdot(f,a)^{-1}>(1,b).$$

Assume now that $g>1$. So we must have $f>\mathscr{C}(g)$, whence $fgf^{-1}>g$ by **GOG2** in $\mathscr{G}$. This implies that $(f,a)\cdot(g,b)\cdot(f,a)^{-1}>(g,b)$. Therefore **GOG2** is satisfied.

We now prove **GOG3**. Let $(g,b)\in\mathscr{G}\ltimes_\rho G$ with $(g,b)\neq(1,0)$ and let $(f,a)\asymp(g,b)$. If $g=1$, then we must have $f=1$ and $a\asymp b$ in G. Given a scaling element $\mathfrak{s}$ in G with $\mathfrak{s}\asymp b$, we see that $(1,\mathfrak{s})$ is scaling in $\mathscr{G}\ltimes_\rho G$ with $(1,\mathfrak{s})\asymp(f,a)$. If $g\neq1$, then we must have $f\asymp g$. Let $\mathfrak{t}\in\mathscr{G}$ be scaling with $\mathfrak{t}\asymp g$ and let $\mathfrak{u}\in\mathscr{C}(\mathfrak{t})$ with $\mathfrak{u}\sim f$. Then $(\mathfrak{u},1)\sim(f,a)$ in $\mathscr{G}\ltimes_\rho G$, which implies that $(\mathfrak{t},1)$ is scaling. So **GOG3** holds in $\mathscr{G}\ltimes_\rho G$.

This shows that $\mathscr{G}\ltimes_\rho G$ is a growth order group. □

Example 3.4 Positive affine maps. Consider an ordered field K and an ordered vector space $(G,+,0,<,.)$ over K. The ordered groups $(K^>,\cdot,1,<)$ and $(G,+,0,<)$ are growth order groups, as they are Abelian. We have an action ρ of $K^>$ on G by scalar multiplication. That is $\rho(\lambda)(a):=\lambda\,.\,a$ for all $\lambda\in K^>$ and $a\in G$. Then $K^>\ltimes_\rho G$ is naturally isomorphic to the group of strictly increasing affine functions $K\longrightarrow G\,;x\mapsto\lambda\,.\,x+a$ for $(\lambda,a)\in K^>\times G$, under composition, and where the ordering is given by

$$(x\mapsto\lambda\,.\,x+a)>x \text{ iff } \lambda\,.\,b+a>b \text{ for sufficiently large } b\in G.$$

The axioms **MGA1** and **MGA2** follow from the fact that $(G,+,0,<,.)$ is an ordered vector space over K. We write $\mathrm{Aff}_K^+(G)$ for the ordered group $K^>\ltimes_\rho G$ given by Theorem 3.2. Since $(G,+,0)$ is Abelian, the axiom **MGA4** is satisfied. Lastly, given $\lambda,\lambda'\in K^>$ with $\lambda>1$ and $a\in G$, we have $\rho(\lambda)(a)-a=\rho(\lambda-1)(a)$ so $\rho(\lambda')(b)=\rho(\lambda)(a)-a$ for $b:=\frac{\lambda-1}{\lambda'}\,.\,a\in G$. In particular **MGA3** holds. Therefore $\mathrm{Aff}_K^+(G)$ is a growth order group.

3.2 Quotients

Given an ordered group $(\mathscr{G},\cdot,1,<)$ and a normal and convex subgroup $N\trianglelefteq\mathscr{G}$, the quotient $\mathscr{G}/N$ is an ordered group (see [16, Section 1.4] or [25, page 260]) for the relation

$$gN<hN\Longleftrightarrow g<h. \tag{8}$$

Lemma 3.5 [25, page 260] *The quotient map $\mathcal{G} \longrightarrow \mathcal{G}/N$ is an ordered group homomorphism.*

The ordering on $\mathcal{G}$ is lexicographic with respect to the orderings on $\mathcal{G}/N$ and N. That is, we have

$$\mathcal{G}^{>} = \{g \in \mathcal{G} : (gN > N) \vee (g \in N^{>})\}. \tag{9}$$

When the short exact sequence $0 \to N \to \mathcal{G} \to \mathcal{G}/N \to 0$ splits, and given a complement H of N in $\mathcal{G}$, we have an ordered group isomorphism $\mathcal{G} \simeq \mathcal{G}/N \rtimes_{\rho} N$ for the morphism $\rho : \mathcal{G}/N \longrightarrow \mathrm{Aut}(N)$ given by

$$\forall g \in \mathcal{G}, \forall f \in N, \rho(gN)(f) := hfh^{-1}$$

for the unique $h \in H \cap gN$, and where $\mathcal{G}/N \rtimes_{\rho} N$ is lexicographically ordered.

We shall now adapt these ideas to the case of growth order groups. If we want both N and $\mathcal{G}/N$ to be growth ordered groups, we have to impose further conditions on $(\mathcal{G}, N)$. This leads to the following definition:

Definition 3.6 *Let $\mathcal{G}$ be a growth order group. A $\preccurlyeq$–initial subgroup of $\mathcal{G}$ is a non-empty subset $N \subseteq \mathcal{G}$ such that for all $f \in N$ and $g \in \mathcal{G}$, we have $g \preccurlyeq f \Longrightarrow g \in N$.*

That an $\preccurlyeq$–initial subgroup is indeed a subgroup follows from Proposition 2.4(1, 2). For the sequel of Section 3.2, we fix a growth order group $(\mathcal{G}, \cdot, 1, <)$ and a normal and $\preccurlyeq$–initial subgroup $N \subseteq \mathcal{G}$.

Proposition 3.7 *Let $H \subseteq \mathcal{G}$ be a $\preccurlyeq$–initial subgroup. Then H is a growth order group which is convex in $\mathcal{G}$.*

Proof That H is convex follows from Theorem 2.6. We note by $\preccurlyeq$–initiality that the centraliser in H of an $h \in H$ is simply its centraliser in $\mathcal{G}$. This is easily seen to imply that H is a growth order group. □

Proposition 3.8 *Assume that the following holds*

$$\forall f, g \in \mathcal{G} \setminus N, [f, g] \in N \Longrightarrow f \asymp g. \tag{10}$$

Then $\mathcal{G}/N$, with the ordering given by (8), is a growth order group.

Proof Let $f, g \in \mathscr{G}$ with $fN > gN > N$. In particular $f, g \in \mathscr{G}^{>}$ and $f > g$. Let $g_0N \in \mathscr{C}(gN)$, so $[g_0, g] \in N$. We have $g_0 \asymp g$ by (10). **GOG1** in $\mathscr{G}$, gives an $f_0 \in \mathscr{C}(f)$ with $f_0 \geqslant g_0$, hence $f_0N \geqslant fN$. We have $[f_0, f] = 1 \in N$ so $f_0N \in \mathscr{C}(fN)$. This shows that **GOG1** holds in $\mathscr{G}/N$.

We next derive **GOG2**. Let $f, g > N$ with $(fN) \succ (gN)$. We have $\mathscr{C}(g)N \subseteq \mathscr{C}(gN)$, so $fN > \mathscr{C}(g)N$, which is equivalent to $f > \mathscr{C}(g)N$. In particular, we have $f > \mathscr{C}(g)$, so $f^{-1} \succ g^{-1}$. By (10), we obtain $[f^{-1}, g^{-1}] \notin N$. But $[f^{-1}, g^{-1}] > 1$ by **GOG2** in $\mathscr{G}$, so $[f^{-1}, g^{-1}] > N$. That is, we have $fgf^{-1}N > gN$, whence **GOG2** holds in $\mathscr{G}/N$.

Finally, let $g \in \mathscr{G} \setminus N$. Let $\mathfrak{s}$ be scaling in $\mathscr{G}$ with $\mathfrak{s} \asymp g$, and let $f \in \mathscr{G}$ with $(fN) \asymp (gN)$ in $\mathscr{G}/N$. From (10), we deduce that there are $g', g'' \in \mathscr{C}(g)$ with $g' \leqslant f \leqslant g''$. This implies that $f \asymp g$, so there is a $\mathfrak{t} \in \mathscr{C}(\mathfrak{s})$ with $\mathfrak{t} \sim f$. We have $\mathfrak{t}N \in \mathscr{C}(\mathfrak{s}N)$ and $(fN)(\mathfrak{t}N)^{-1} = (f\mathfrak{t}^{-1})N \prec fN$. We claim that $\mathscr{C}(\mathfrak{s}N) = \{\mathfrak{u}N : \mathfrak{u} \in \mathscr{C}(\mathfrak{s})\}$. Indeed, let $g \in \mathscr{G}$ with $gN \in \mathscr{C}(\mathfrak{s}N)$, so $[g, \mathfrak{s}] \in N$. We have $g \asymp \mathfrak{s}$ so there is a $\mathfrak{t} \in \mathscr{C}(\mathfrak{s})$ with $g \sim \mathfrak{t}$. Writing $\delta := \mathfrak{t}^{-1}g$, we have $[g, \mathfrak{s}] = \delta^{-1}\mathfrak{t}^{-1}\mathfrak{s}^{-1}\mathfrak{t}\delta\mathfrak{s} = [\mathfrak{t}\delta, \mathfrak{s}]$. Since $\mathfrak{t}$ and $\mathfrak{s}$ commute, we obtain $[g, \mathfrak{s}] = [\delta, \mathfrak{s}] \in N$. As $\delta \prec \mathfrak{s}$, we deduce with (10) that $\delta \in N$, so $gN = \mathfrak{t}N$ as claimed. Recall by Corollary 2.21 that $\mathscr{C}(\mathfrak{s})$ is Abelian. Thus $\{\mathfrak{u}N : \mathfrak{u} \in \mathscr{C}(\mathfrak{s})\}$ is Abelian, and $\mathfrak{s}N$ is scaling in $\mathscr{G}/N$. So **GOG3** holds. □

Given two linearly ordered sets $(A, <)$ and $(B, <)$, we write $A \amalg B$ for the disjoint union $A \times \{0\} \sqcup B \times \{1\}$ ordered so that $A \times \{0\} < B \times \{1\}$ and that $a \mapsto (a, 0)$ and $b \mapsto (b, 1)$ are ordered embeddings $A \longrightarrow A \amalg B$ and $B \longrightarrow A \amalg B$ respectively. In the next proof, we use the notation $v_{\mathscr{G}}$ for the standard valuation on a growth order group $\mathscr{G}$, in order to distinguish between various growth order groups.

Proposition 3.9 *Assume that* (10) *holds. We have an isomorphism of ordered sets*

$$\Phi : v_{\mathscr{G}}(\mathscr{G}) \longrightarrow v_N(N) \amalg v_{\mathscr{G}/N}(\mathscr{G}/N)$$

defined by $\Phi(v(g)) := v(gN)$ *if* $g \notin N$ *and* $\Phi(v(g)) = v_N(g)$ *if* $g \in N^{\neq}$.

Proof Let $g, h \in \mathscr{G}^{\neq}$ with $g \asymp h$. If $g \notin N$, then we have $g \succ N$, and $hN \asymp gN$ by Theorem 3.5, whence $\Phi(v_{\mathscr{G}}(g))$ is well defined. If $g \in N$, then $h \in N$. Since N is $\preccurlyeq$–initial, we have $v_N(g) = v_{\mathscr{G}}(g) = v_{\mathscr{G}}(h) = v_N(h)$, so Φ is well-defined. It is clear that Φ is surjective.

Now let $f, g \in \mathscr{G}$ with $f \prec g$. We want to prove that $\Phi(v_{\mathscr{G}}(f)) < \Phi(v_{\mathscr{G}}(g))$. If $f \notin N$ and $g \notin N$, then $v(fN) < v(gN)$ by Theorem 3.5. So

$$\Phi(v_{\mathscr{G}}(f)) = v_{\mathscr{G}/N}(f) < v_{\mathscr{G}/N}(g) = \Phi(v_{\mathscr{G}}(g)).$$

If $f \in N$ and $g \notin N$, then $f \prec g$ and $\Phi(v_{\mathscr{G}}(f)) < \Phi(v_{\mathscr{G}}(g))$ by definition. If $f, g \in N$, then $\Phi(v_{\mathscr{G}}(f)) = v_N(f) < v_N(g) = \Phi(v_{\mathscr{G}}(g))$. This concludes the proof. □

3.3 Growth order groups of finite value set

We fix a non-trivial growth order group $\mathscr{G}$ such that $v(\mathscr{G})$ has a maximal element $v(f_0)$. Let $\mathfrak{s}$ be scaling with $\mathfrak{s} \asymp f_0$. Write $\mathscr{G}^{\prec \mathfrak{s}} := \{g \in \mathscr{G} : g \prec \mathfrak{s}\}$. Note that $\mathscr{G}^{\prec \mathfrak{s}} \neq \varnothing$.

Proposition 3.10 *The set $\mathscr{G}^{\prec \mathfrak{s}}$ is a normal and $\preccurlyeq$–initial subgroup of $\mathscr{G}$.*

Proof This set is $\preccurlyeq$–initial by definition. It is normal by Theorem 2.9. □

By Theorem 3.7, the subgroup $\mathscr{G}^{\prec \mathfrak{s}}$ is a growth order group.

Proposition 3.11 *The subgroup $\mathscr{C}(\mathfrak{s})$ is a complement of $\mathscr{G}^{\prec \mathfrak{s}}$.*

Proof We have $\mathscr{G}^{\prec \mathfrak{s}} \cap \mathscr{C}(\mathfrak{s}) = \{t \in \mathscr{C}(\mathfrak{s}) : t \prec \mathfrak{s}\} = \{1\}$. For $g \in \mathscr{G}$, we either have $g \prec \mathfrak{s}$, and then $g \in \mathscr{G}^{\prec \mathfrak{s}}$, or $g \asymp \mathfrak{s}$, and then given $t \in \mathscr{C}(\mathfrak{s})$ with $t \sim g$, we have $gt^{-1} \prec \mathfrak{s}$, whence $g = (gt^{-1})t \in \mathscr{G}^{\prec \mathfrak{s}} \mathscr{C}(\mathfrak{s})$. □

Thus the sequence $0 \to \mathscr{G}^{\prec \mathfrak{s}} \to \mathscr{G} \to \mathscr{G}/\mathscr{G}^{\prec \mathfrak{s}} \to 0$ splits, and we have a natural isomorphism $\mathscr{G}^{\prec \mathfrak{s}} \rtimes \mathscr{C}(\mathfrak{s}) \longrightarrow \mathscr{G}$. If follows by induction that if $v(\mathscr{G}) = \{\rho_1, \ldots, \rho_n\}$ is finite with $\rho_1 > \cdots > \rho_n$, then $\mathscr{G}$ is an iterated semidirect product

$$\mathscr{G} \simeq (\cdots (\mathscr{C}_{\rho_n} \rtimes \mathscr{C}_{\rho_{n-1}}) \rtimes \cdots) \rtimes \mathscr{C}_{\rho_1}. \tag{11}$$

This can be taken as a conclusion to our discussion in Section 2.6 in the case of finite value set, ie a positive answer to Question 1 in that case.

Proposition 3.12 *Suppose that $\mathscr{G}$ has value set $n > 0$ and let t be scaling with $v(t) = \min v(\mathscr{G}^{\neq})$. If $\mathscr{C}(t)$ is Archimedean, then $n \leqslant 2$.*

Proof Assume for contradiction that $n > 2$. Using the above decomposition $n - 3$ times, we may assume that $n = 3$. Fix two scaling elements $\mathfrak{s}_1, \mathfrak{s}_2$ with $t \prec \mathfrak{s}_1 \prec \mathfrak{s}_2$. So $\mathscr{G} \simeq \mathscr{G}_1 \rtimes \mathscr{C}(\mathfrak{s}_2)$ where $\mathscr{G}_1 = \mathscr{C}(t) \rtimes \mathscr{C}(\mathfrak{s}_1)$. Let $\sigma \in \mathrm{Aut}(\mathscr{G}_1)$ be the conjugation by $\mathfrak{s}_2$ and let $\chi \in \mathrm{Aut}(\mathscr{C}(t))$ be the conjugation by $\mathfrak{s}_1$. Since $t \prec \mathfrak{s}_1$ in $\mathscr{G}_1$, we have $\sigma(t) \prec \sigma(\mathfrak{s}_1)$, whence $\sigma(t) \asymp t$. But then $\sigma(t) \in \mathscr{C}(t)$. For $n \in \mathbb{N}$ we have $\mathfrak{s}_1^n \prec \mathfrak{s}_2$, so $\sigma(t) > \chi^{[n]}(t)$ by **GOG2**. Since $\mathscr{C}(t)$ is Archimedean, this contradicts [30, Theorem 1.5.1]. □

3.4 O-minimal germs

Let $\mathcal{M} = (M, \ldots)$ be a first-order structure in a language $\mathcal{L}$. Assume that $\mathcal{M}$ has definable Skolem functions (allowing parameters). This is the case for instance if $\mathcal{M}$ is an o-minimal expansion of an ordered group in a language expanding $\mathcal{L}_{\mathrm{og}}$.

Let $n > 0$ and let p be an n–type in $\mathcal{M}$ over M whose finite subsets are realised in $\mathcal{M}$. Let $p(\mathcal{M}) := \{\varphi(M^n) : \varphi \in p\}$ be the corresponding ultrafilter on the Boolean algebra of definable subsets of M^n. Consider the set $\mathcal{F}_n$ of functions $M^n \longrightarrow M$ that are definable in $\mathcal{M}$ with parameters, and the set $\mathcal{M}_p$ of germs at p

$$[f]_p := \{g \in \mathcal{F}_n : \exists X \in p(\mathcal{M}), f \text{ and } g \text{ coincide on } X\}$$

of such functions. If R is a relation symbol of arity $k \in \mathbb{N}$ in the corresponding language (including function symbols and constant symbols), then R is interpreted on $\mathcal{M}_p$ as the well-defined subset of tuples $([f_1], \ldots, [f_k])$ for which there is an $X \in p(\mathcal{M})$ with $\mathcal{M} \models R\,[f_1\,(\overline{m})\,, \ldots, f_n\,(\overline{m})]$ for all $\overline{m} \in X$.

It is a folklore result that $\mathcal{M}_p$ is an elementary extension of $\mathcal{M}$ for the natural inclusion $\Psi : \mathcal{M} \longrightarrow \mathcal{M}_p$ sending $\overline{m_0} \in M$ to the germ of the constant function $\overline{m} \mapsto \overline{m_0}$. This follows from the following lemma:

Lemma 3.13 *For all $\mathcal{L}$–formulas $\varphi(v_1, \ldots, v_k)$ with parameters in M and $f_1, \ldots, f_k \in \mathcal{F}_n$, we have*

$$\{\overline{m} \in M^n : \mathcal{M} \models \varphi(f_1(\overline{m}), \ldots, f_p(\overline{m}))\} \in p(\mathcal{M})$$

if and only if $\mathcal{M}_p \models \varphi([f_1], \ldots, [f_k])$.

Suppose that $\mathcal{L}$ contains a binary relation symbol $<$ and that $\mathcal{M} = (M, <, \ldots)$ is o-minimal. The set of formulas $m < v_0$, in one free variable v_0, where m ranges in M induces a unique type p_∞ over M called the type at infinity. The germ $[f]$ at p_∞ of an $f \in \mathcal{F}_n$ is simply its germ at $+\infty$. We write $\mathcal{M}_\infty := \mathcal{M}_{p_\infty}$. The ordering on $\mathcal{M}_\infty$ is given by $[f_0] < [f_1] \Longleftrightarrow f_0(m) < f_1(m)$ for all sufficiently large $m \in M$. By the monotonicity theorem [12, Chapter 3, (1.2)] a definable function $f : M \longrightarrow M$ is strictly monotonic on some neighborhood of $+\infty$. A germ $[f]$ lies above each $m \in M$ under the embedding $\mathcal{M} \longrightarrow \mathcal{M}_\infty$ if and only if f tends to $+\infty$ at $+\infty$. We define $\mathcal{G}_{\mathcal{M}}$ as the subset of $\mathcal{M}_\infty$ of germs $[f]$ with $[f] > M$. A germ in $\mathcal{G}_{\mathcal{M}}$ cannot be constant or strictly decreasing, so it is strictly increasing. We write id for the identity function on M, so $[\mathrm{id}] \in \mathcal{G}_{\mathcal{M}}$.

Since $\mathcal{M}$ is o-minimal, for any $[f], [g] \in \mathcal{F}_\infty$, there is an $m \in M$ such that $f((m, +\infty))$ is a neighbourhood of $+\infty$. We may choose m so that $f((m, +\infty)) = (f(m), +\infty)$.

So f induces a strictly increasing bijection between two neighbourhoods of $+\infty$. The germ of $f \circ g$ lies in $\mathcal{G}_{\mathcal{M}}$. Since this germ does not depend on f, g we may define $[f] \circ [g] := [f \circ g]$. Note that $[f] \circ [\mathrm{id}] = [\mathrm{id}] \circ [f] = [f]$. Writing f^{inv} for the inverse of $f : (m, +\infty) \longrightarrow (f(m), +\infty)$, we see that $[f^{\mathrm{inv}}]$ only depends on $[f]$, and we have $[f] \circ [f^{\mathrm{inv}}] = [f^{\mathrm{inv}}] \circ [f] = [\mathrm{id}]$. Thus $(\mathcal{G}_{\mathcal{M}}, \circ, [\mathrm{id}])$ is a group. The ordering on $\mathcal{G}_{\mathcal{M}}$ induced by that on $\mathcal{M}_{\infty}$ is a left-ordering because the germs are strictly increasing. It is a right-ordering by definition. So $(\mathcal{G}_{\mathcal{M}}, \circ, [\mathrm{id}], <)$ is an ordered group.

This raises the naive question: is $\mathcal{G}_{\mathcal{M}}$ always a growth order group? The answer is negative. Indeed, it is known [8, Theorem 8] that given any ordered group $(\mathcal{G}, \cdot, 1, <)$, the structure $\mathcal{M} := (\mathcal{G}, <, (t_g)_{g \in \mathcal{G}})$ where each t_g for $g \in \mathcal{G}$ is the unary function $\mathcal{G} \longrightarrow \mathcal{G} \,; h \mapsto gh$ eliminates quantifiers and has a universal axiomatisation. In particular, it is o-minimal, and $g \mapsto [t_g]$ is an isomorphism between $(\mathcal{G}, \cdot, 1, <)$ and $(\mathcal{G}_{\mathcal{M}}, \circ, [\mathrm{id}], <)$. If $\mathcal{G}$ is not a growth order group, then neither is $\mathcal{G}_{\mathcal{M}}$. We may still ask whether $\mathcal{G}_{\mathcal{M}}$ is a growth order group when $\mathcal{M}$ expands the real ordered field. We will answer this question in the positive in a particular case in the next section. We finish with a positive answer to the naive question for pure ordered groups:

Example 3.14 Let $\mathcal{M} := (G, +, 0, <)$ be a non-trivial o-minimal ordered group. This is a divisible, Abelian ordered group [33], so it has Skolem functions. Recall [34] that the $\mathcal{L}_{\mathrm{og}}$–theory T_{daog} of non-trivial divisible Abelian ordered group is complete and has quantifier elimination in $\mathcal{L}_{\mathrm{og}}$. It has a universal axiomatisation in the language $\mathcal{L}_{\mathrm{doag}} := \langle \cdot, 1, <, \mathrm{Inv}, (\mu_q)_{q \in \mathbb{Q}} \rangle$ where each $\mu_q, q \in \mathbb{Q}$ is interpreted as the scalar multiplication $x \mapsto q \,.\, x$. This implies that the germ at $+\infty$ of each definable function $G \longrightarrow G$ is that of a term in $\mathcal{L}_{\mathrm{doag}}$. So each element of $\mathcal{M}_{\infty}$ is the germ of

$$G \longrightarrow G \,; x \mapsto q \,.\, x + y$$

for fixed $q \in \mathbb{Q}$ and $y \in G$. In other words $\mathcal{G}_{\mathcal{M}}$ is isomorphic to the growth order group $\mathrm{Aff}^{+}_{\mathbb{Q}}(G)$ of Theorem 3.4.

4 H-fields with composition and inversion

An *H-field* [1, 2] is an ordered valued field $(K, +, \times, 0, 1, <, \mathcal{O})$ with convex valuation ring $\mathcal{O}$ and maximal ideal thereof $\mathcal{o}$, equipped with a derivation $\partial : K \longrightarrow K$ such that the following conditions are satisfied:

HF1 $\forall a \in \mathcal{O}, \exists c \in \mathrm{Ker}(\partial), a - c \in \mathcal{o}$.

HF2 $\forall a \in K, a > \mathrm{Ker}(\partial) \Longrightarrow \partial(a) > 0$.

We usually denote Ker(∂) by C. This is a subfield of K called the *field of constants*. We write $K^{>C} := \{a \in K : \forall c \in C, a > c\}$. For $a \in K$, we often write $a' = \partial(a)$, and we use the Landau notations $\mathcal{O}(a) := \mathcal{O}a = \{\delta a : \delta \in \mathcal{O}\}$ and $o(a) := o a = \{\varepsilon a : \varepsilon \in o\}$. So $\mathcal{O}(1) = \mathcal{O}$ and $o(1) = o$. For $a \in K^\times$, we write

$$a^\dagger := \frac{a'}{a} \in K.$$

Note that $(ab)^\dagger = a^\dagger + b^\dagger$ and $(ca)^\dagger = a^\dagger$ for all $b \in K^\times$ and $c \in C^\times$. We have the following important valuative inequality [1, Lemma 1.1]:

$$\forall a, b \in o, b' \in o(a^\dagger). \tag{12}$$

Furthermore, we have [37, Corollary 1] l'Hospital's rule

$$\forall f, g \in \mathcal{H}, ((f \in o(g) \wedge g \notin \Theta(1)) \Longrightarrow f' \in o(g')). \tag{13}$$

4.1 H-fields with composition

We now expand H-fields with a composition law.

Definition 4.1 *An **H-field with composition** (over $C = \mathrm{Ker}(\partial)$) is an H-field $(K, +, \cdot, 0, 1, <, \mathcal{O}, \partial)$ with a fixed $x \in K^{>C}$ such that $x' = 1$, and a binary operation $\circ : K \times K^{>C} \longrightarrow K$ satisfying the following conditions:*

HFC1 *For all $b \in K^{>C}$, the function $K \longrightarrow K$; $a \mapsto a \circ b$ is a C–linear morphism of ordered rings.*

HFC2 *For all $a \in K$ and $b, d \in K^{>C}$, we have $a \circ (b \circ d) = (a \circ b) \circ d$.*

HFC3 *For all $a \in K^{>C}$, the function $K^{>C} \longrightarrow K^{>C}$; $b \mapsto a \circ b$ is strictly increasing.*

HFC4 *For all $a \in K$ and $b \in K^{>C}$, we have*

$$a \circ x = a \qquad \text{and} \qquad x \circ b = b.$$

HFC5 *Let $a, \delta \in K$ and $b \in K^{>C}$ with $\delta \in o(b)$ and $(a^\dagger \circ b)\delta \in o$. For all $n \in \mathbb{N}$, we have*

$$a \circ (b + \delta) - \sum_{k \leqslant n} \frac{a^{(k)} \circ b}{k!} \delta^k \in o((a^{(n)} \circ b)\delta^n),$$

where $a^{(k)}$ denotes the k–th derivative of a.

Consider the language $\mathscr{L}_{\mathrm{hfc}}$ expanding the language of ordered valued differential fields with a constant symbol $\underline{x}$ and a binary function symbol $\circ$. We interpret $\underline{x}$ on K as expected and extend $\circ$ to $K \times K$ by setting $a \circ b := 0$ if $b \notin K^{>C}$. Thus K is an $\mathscr{L}_{\mathrm{hfc}}$–structure, and the class of H-fields with composition is elementary in $\mathscr{L}_{\mathrm{hfc}}$.

The axioms **HFC1**–**HFC4** imply that $(K^{>C}, \circ, x, <)$ is an ordered monoid that acts by automorphisms on $(K, +, \cdot, 0, 1, <, \mathcal{O})$, by post-composition. In order to avoid confusion between compositions and products in K, given an $a \in K^{>C}$ and an $n \in \mathbb{N}$, we write $a^{[n]}$ for the n–fold iterate of a (ie its n–th power in the monoid $K^{>C}$). If a has an inverse in $K^{>C}$, then we denote it by a^{inv} and we set $a^{[-n]} := (a^{\mathrm{inv}})^{[n]} = (a^{[n]})^{\mathrm{inv}}$.

Example 4.2 Let C be an ordered field. Let $C(x)$ be a purely transcendental simple extension, ordered so that $x > C$. Write $\mathcal{O}$ for the convex hull of C in $C(x)$, which is the set of fractions with degree ≤ 0.

We have a derivation $\partial : C(x) \longrightarrow C(x)$ with respect to x, which is determined by $C = \mathrm{Ker}(\partial)$ and $\partial(x) = 1$. And $(C(x), +, \cdot, 0, 1, \partial, \mathcal{O}, <)$ is an H-field. For $P \in C(x)$ and $Q \in C(x)^{>C}$, since Q lies above each pole of P, the compositum $P \circ Q$ is well-defined. It is easy to see that **HFC1**–**HFC4** are satisfied. Les us now justify that **HFC5** holds. Let $F \in C(x)$ and b, δ, n as in **HFC5**. We have $F' \in \mathcal{O}x^{-1}F$, so $(F^{(k+1)} \circ b)\delta^{k+1} \in \mathcal{O}(F^{(k)} \circ b)\delta^k \frac{\delta}{b} \subseteq \mathcal{o}(F^{(k)} \circ b)\delta^k$ for each $k \in \mathbb{N}$. We have formal identity $F \circ (b + y) = \sum_{k \in \mathbb{N}} \frac{F^{(k)} \circ b}{k!} y^k$ in $C[[x, y]]$, and the previous argument entails that plugging δ for y gives a convergent sum for the valuation topology on $C[[x]]$. It also entails that $F \circ (b + \delta) - \sum_{k \leqslant n} \frac{F^{(k)} \circ b}{k!} \delta^k = \sum_{k > n} \frac{F^{(k)} \circ b}{k!} \delta^k \in \mathcal{O}((F^{(n+1)} \circ b)\delta^{n+1}) \subseteq \mathcal{o}((F^{(n)} \circ b)\delta^n)$.

Note that each H-field with composition over C contains $C(x)$ as an $\mathscr{L}_{\mathrm{hfc}}$–substructure.

Example 4.3 Consider the field $\mathbb{T}_g$ of grid-based transseries [14, 20]. We have a derivation and composition law [21] on $\mathbb{T}_g$ such that it is an H-field with field of constants $\mathbb{R}$ and that **HFC1**, **HFC2**, **HFC4** and **HFC5** are satisfied. As for **HFC3**, it follows from the inclusion of $\mathbb{T}_g$ in the field of finitely nested hyperseries of [5], where it holds. By [21, Section 5.4], this field has inversion.

We will see other, more analytic examples in the next section (see Theorem 4.16). We now state a few simple consequences of the axioms.

Remark 4 If $\varepsilon \in \mathcal{o}$, then $\varepsilon' \in \mathcal{o}((x^{-1})^\dagger) = \mathcal{o}(x^{-1})$ by (12). In particular $\varepsilon' \in \mathcal{o}$, so the derivation on K is small as per [3, page 7].

As an ordered field, any H-field has a field topology, called the order topology, for which the family of $(-\varepsilon, \varepsilon), \varepsilon \in K^{>}$ is a fundamental system of neighbourhoods of 0. We understand limits in that sense.

Lemma 4.4 *Let K be an H-field with composition. For $a \in K$ and $b \in K^{>C}$, we have*

$$a' \circ b = \lim_{\substack{\delta \to 0 \\ \delta \neq 0}} \frac{a \circ (b+\delta) - a \circ b}{\delta}.$$

Proof Let $\delta \in K$ be sufficiently small in absolute value, so that $\delta \in \prec\!\!\!\!\circ(b)$ and $(a^{\dagger} \circ b)\delta \in \prec\!\!\!\!\circ$. By **HFC5** for $n = 1$, we have $a \circ (b+\delta) - a \circ b - (a' \circ b)\delta \in \prec\!\!\!\!\circ((a'' \circ b)\delta^2)$, so

$$\left| \frac{a \circ (b+\delta) - a \circ b}{\delta} - (a' \circ b)\delta \right| < |(a'' \circ b)\delta|.$$

Letting δ tend to 0, we obtain the desired result. □

Lemma 4.5 *Let K be an H-field with composition. Let $a \in K$ and $b \in K^{>C}$. We have*

$$(a \circ b)' = (a' \circ b)b'.$$

Proof Write $\tau(\delta) := \delta^{-1}((a \circ b) \circ (x+\delta) - a \circ b)$ for all $\delta \neq 0$, so

$$(a \circ b)' = \lim_{\substack{\delta \to 0 \\ \delta \neq 0}} \tau(\delta).$$

By **HFC5** for (b, x, δ), we can have $b \circ (x+\delta) - b \in b'\delta + \mathcal{O}(b''\delta^2)$ arbitrarily small by choosing δ small enough. In turn, applying **HFC5** for $(a, b, b \circ (x+\delta) - b)$ we obtain $\delta\tau(\delta) - (a' \circ b)b'\delta \in \mathcal{O}((a' \circ b)b''\delta^2)$ provided δ is sufficiently small. We thus have $\tau(\delta) - (a' \circ b)b' \in \mathcal{O}((a' \circ b)b''\delta)$, hence the result. □

We say that K is an *H-field with composition and inversion* if furthermore $(K^{>C}, \circ, x)$ is a group. Then in view of **HFC1–HFC4**, the structure $(K^{>C}, \circ, x, <)$ is an ordered group. We will give conditions for it to be a growth order group. More precisely, consider the following conditions on an ordered pair $(\mathcal{G}_0, \mathcal{G}_1)$ of subgroups of $K^{>C}$:

($\star$) The subset $\mathcal{G}_0 \subseteq K^{>C}$ is a normal convex subgroup of $K^{>C}$ containing $x + C$, the subset $\mathcal{G}_1 \subseteq K^{>C}$ is a complement of $\mathcal{G}_0$ in $K^{>C}$ which is a growth order group with Archimedean centralisers, and $\{a \circ (a^{\mathrm{inv}} + 1) : a \in \mathcal{G}_1\}$ is cofinal in $\mathcal{G}_0$.

We will obtain Theorem 2 as a consequence of the following theorem.

Theorem 4.6 *Let $(K, +, \cdot, 0, 1, \partial, \mathcal{O}, <, \circ, x)$ be an H-field over $\mathbb{R}$ with composition and inversion and let $(\mathcal{G}_0, \mathcal{G}_1)$ be as in ($\star$). Then $K^{>\mathbb{R}}$ is a growth order group with Archimedean centralisers, and $\mathcal{G}_0$ is a growth order group which is $\preccurlyeq$–initial in $K^{>\mathbb{R}}$.*

This will be proved in Section 4.4 below.

4.2 Taylor approximations in Hardy fields

Let $\mathscr{C}^{<\infty}$ denote the set of all germs $[f]$ at $+\infty$ of real-valued functions f defined on positive half-lines $(a,+\infty), a \in \mathbb{R}$ such that for each $k \in \mathbb{N}$, there is a positive half-line on which f is k–times differentiable. We identify constants with the germs of the corresponding constant functions. Then $\mathscr{C}^{<\infty}$ is an $\mathbb{R}$–algebra under pointwise sum and product. Moreover, it is equipped with a partial $\mathbb{R}$–algebra ordering given by $[f] < [g]$ if and only $f(t) < g(t)$ for all sufficiently large $t \in \mathbb{R}$. It is a differential ring under derivation of germs $[f]' := [f']$ whenever $f : (a,+\infty) \longrightarrow \mathbb{R}$ is differentiable. Finally, if $[g] > \mathbb{R}$ in $\mathscr{C}^{<\infty}$, ie if g tends to $+\infty$ at $+\infty$, then for all $[f] \in \mathscr{C}^{<\infty}$ where $f \circ g$ is defined on a positive half-line, the germ $[f] \circ [g] := [f \circ g]$ only depends on $[f]$ and $[g]$.

We will identify germs with given representatives, trying not to confuse the reader in the process. Given a germ $g \in \mathscr{C}^{<\infty}$, we write

$$\begin{aligned} \prec(g) &:= \{f \in \mathscr{C}^{<\infty} : \forall r \in \mathbb{R}^{>}, |f| < r|g|\} \\ \mathcal{O}(g) &:= \{f \in \mathscr{C}^{<\infty} : \exists r \in \mathbb{R}, |f| < r|g|\}, \quad \text{and} \\ \Theta(g) &:= \{f \in \mathscr{C}^{<\infty} : f \in \mathcal{O}(g) \wedge g \in \mathcal{O}(f)\}. \end{aligned}$$

We simply write $\prec, \mathcal{O}$ and Θ for $\prec(1), \mathcal{O}(1)$ and $\Theta(1)$ respectively.

Recall that a *Hardy field* is a differential subfield of $\mathscr{C}^{<\infty}$ containing all constant germs. The induced ordering on such fields is linear [10, page 107].

Definition 4.7 *A* **Hardy field with composition** *is a Hardy field $\mathscr{H}$ which is closed under composition of germs. We say that it has inversion if $\mathscr{H}^{>\mathbb{R}}$ is closed under inversion.*

Example 4.8 If $\mathscr{R}$ is an o-minimal expansion of the real ordered field, then $\mathscr{R}_\infty$ is a Hardy field [12, Section 7.1] with composition and inversion.

Example 4.9 The intersection of all $\subseteq$–maximal Hardy fields is a Hardy field with composition [9]. It is unknown whether it has inversion.

We will show that certain Hardy fields with composition and inversion are H-fields with composition and inversion. This mainly entails deriving the Taylor axiom **HFC5** in those fields. If $\mathscr{H}$ is a Hardy field, then $\mathcal{O}(1) \cap \mathscr{H}$ is a valuation ring on $\mathscr{H}$ for which it is an H-field. The notations above are consistent with that introduced for H-fields. The derivation on $\mathscr{H}$ is *small*, ie

$$\prec' \subseteq \prec. \tag{14}$$

See [38, Section 2] or [3, Proposition 9.1.9] for proofs. Toward proving **HFC5**, we need a mean value theorem for germs.

Lemma 4.10 *Let $\mathcal{H}$ be a Hardy field with composition and inversion and let $f \in \mathcal{H}$ and $g, h \in \mathcal{H}^{>}$ with $g < h$. There is a $c \in \mathcal{H}$ with $g < c < h$ and $f \circ h - f \circ g = (h-g)f' \circ c$.*

Proof Assume first that $f' \in \mathbb{R}$. So f is the germ of an affine function $f = a\,\mathrm{id} + b$, and we have $f \circ h - f \circ g = (h-g)a = (h-g)f' \circ c$ where $c := \frac{g+h}{2} \in (g, h)$.

Assume now that $f' \notin \mathbb{R}$. So f' is the germ of a strictly monotonic function. Let $t \in \mathbb{R}$ be large enough so that $h(s) > g(s)$ for all $s \geqslant t$, that f is differentiable on $[t, +\infty)$ and that f' is strictly monotonic on $[t, +\infty)$. The mean value theorem for f gives $f(h(t)) - f(g(t)) = (h(t) - g(t))f'(c_t)$ for a certain $c_t \in (g(t), h(t))$. Since f' is strictly monotonic on $[t, +\infty)$, the number c_t is unique, and we have a function $t \mapsto c_t$ whose germ c satisfies $c \in (g, h)$ and $f \circ h - f \circ g = (h-g)f' \circ c$.

Note that $f' \circ c = \frac{f \circ h - f \circ g}{h-g} \in \mathcal{H}$. Our hypothesis that $f' \notin \mathbb{R}$ means that f' is the germ of a strictly monotonic function, which thus induces a bijection $\varphi : (t_0, +\infty) \longrightarrow (t_1, t_2)$ for some $t_0 \geqslant t$ and $t_1, t_2 \in \mathbb{R} \cup \{\pm\infty\}$ with $t_1 < t_2$. By considering translations, homotheties and inversions if necessary, we may assume that $t_2 = +\infty$, so

$$c = \varphi^{\mathrm{inv}} \circ \frac{f \circ h - f \circ g}{h - g}$$

lies in $\mathcal{H}$. □

Lemma 4.11 *For all $f \in \mathcal{H}^{>\mathbb{R}}$ with $f^\dagger \in \mathcal{O}(\mathrm{id}^{-1})$, we have $(f')^\dagger \in \mathcal{O}(\mathrm{id}^{-1})$.*

Proof We have $f'\,\mathrm{id} \in \mathcal{O}(f)$ where $f \notin \Theta(1)$, so $(f''\,\mathrm{id} + f') \in \mathcal{O}(f')$ by (13). We recall that $\mathcal{O}$ is a valuation ring on $\mathcal{H}$. Since $f' \in \mathcal{O}(f')$, we must have $f''\,\mathrm{id} \in \mathcal{O}(f')$, ie $(f')^\dagger \in \mathcal{O}(\mathrm{id}^{-1})$. □

Lemma 4.12 *For all $f \in \mathcal{H}^{>\mathbb{R}}$ with $f^\dagger \notin \mathcal{O}(\mathrm{id}^{-1})$, we have $(f')^\dagger \in \mathcal{O}(f^\dagger)$.*

Proof By [38, Theorem 2], there is a Hardy field $\mathcal{H}^*$ containing $\mathcal{H}$ and which is closed under composition on the left of strictly positive germs with the germ log of the natural logarithm. Note that $f^\dagger = (\log \circ f)'$. We have $\mathrm{id}^{-1} \in o(f^\dagger)$ in $\mathcal{H}^*$ ie $\log' \in o((\log \circ f)')$. So (13) gives $\log \in o(\log \circ f)$. This means that $\mathbb{N} \log < \log \circ f$, so $\mathrm{id}^{\mathbb{N}} < f$. In particular $\mathrm{id}^2 \in o(f)$ so (13) yields $2\,\mathrm{id} \in o(f')$, whence $2 < f''$ by **HF2**. Now $-\frac{f'}{f^2} = (f^{-1})' \in o$ by (14), which means that $f' \in o(f^2)$. We deduce with [1, Lemma 1.4] that $(f')^\dagger < (f^2)^\dagger = 2f^\dagger$. Since $f'' > 0$ and $f' > 0$, we have $(f')^\dagger > 0$, so this entails that $(f')^\dagger \in \mathcal{O}(f^\dagger)$. □

Proposition 4.13 *Let $\mathcal{H}$ be a Hardy field with composition. Then $\mathcal{H}$ satisfies* **HFC5** *if and only if for all $f, g \in \mathcal{H}^{>\mathbb{R}}$ and $\delta \in \prec(g)$ with $(f^\dagger \circ g)\delta \in \prec$, we have*

$$f \circ (g + \delta) \in \Theta(f \circ g). \tag{15}$$

Proof The relation in (15) is implied by **HFC5** at $n = 0$. Assume that (15) holds. Let f, g, δ be as in the statement of the proposition. We claim that

$$\forall n \in \mathbb{N}, (f^{(n+1)} \circ g)\delta^{n+1} \in \prec((f^{(n)} \circ g)\delta^n). \tag{16}$$

Indeed, for $n = 0$, this follows from the assumption on δ. Let $n \in \mathbb{N}$ such that (16) holds at n. Suppose that $(f^{(n)})^\dagger \in \mathcal{O}(\mathrm{id}^{-1})$. Then $(f^{(n+1)})^\dagger \in \mathcal{O}(\mathrm{id}^{-1})$ by Theorem 4.11, ie $f^{(n+2)} \in \mathcal{O}(\mathrm{id}^{-1} f^{(n+1)})$. Composing with g and then multiplying by δ^{n+2}, we obtain

$$(f^{(n+2)} \circ g)\delta^{n+2} \in \frac{\delta}{g}\mathcal{O}((f^{(n+1)} \circ g)\delta^{n+1}).$$

But $\delta \in \prec(g)$ so $(f^{(n+2)} \circ g)\delta^{n+2} \in \prec((f^{(n+1)} \circ g)\delta^{n+1})$ as claimed. Suppose now that $(f^{(n)})^\dagger \notin \mathcal{O}(\mathrm{id}^{-1})$. Then Theorem 4.12 gives

$$(f^{(n+1)})^\dagger \in \mathcal{O}((f^{(n)})^\dagger)$$

so $((f^{(n+1)})^\dagger \circ g)\delta \in \mathcal{O}(((f^{(n)})^\dagger \circ g)\delta) \subseteq \prec$ by the induction hypothesis. Therefore $(f^{(n+2)} \circ g)\delta\delta^{n+1} \in \prec((f^{(n+1)} \circ g)\delta^{n+1})$. We conclude by induction that (16) holds.

Let us now derive **HFC5** at a given $n \in \mathbb{N}$. Suppose $\delta \geqslant 0$. Let $r_0, r_1 \in \mathbb{R}^>$ and let $t_0 \in \mathbb{R}$ be large enough so that f is $\mathscr{C}^{n+1}$ on $[t_0, +\infty)$, that δ is non-negative on $[t_0, +\infty)$, and that $f^{(n+1)}$ is monotonic on $g(t_0, +\infty)$. By (15) for $f^{(n+1)}$, we may also assume that

$$r_0|f^{(n+1)}(g(t))| \leqslant |f^{(n+1)}(g(t) + \delta(t))| \leqslant r_1|f^{(n+1)}(g(t))|$$

for all $t \in (t_0, +\infty)$. By Taylor's theorem, for $t > t_0$, the integral

$$I(t) := \int_{g(t)}^{g(t)+\delta(t)} \frac{(g(t) + \delta(t) - s)^n}{n!} f^{(n+1)}(s)ds$$

satisfies $f(g(t) + \delta(t)) = \sum_{k=0}^n \frac{f^{(k)}(g(t))}{k!}\delta(t)^k + I(t)$. Now $|I(t)|$ is bounded by the integral

$$\int_{g(t)}^{g(t)+\delta(t)} \frac{(g(t) + \delta(t) - s)^n}{n!} r_1|f^{(n+1)}(g(t))|ds = \frac{|f^{(n+1)}(g(t))|}{(n+1)!}\delta(t)^{n+1}.$$

Thus **HFC5** at n follows from (16). The case when $\delta \leqslant 0$ is similar. □

Lemma 4.14 *Let $\mathcal{H}$ be a Hardy field with composition and inversion. Then for $x = \mathrm{id}$, the axioms* **HFC1**–**HFC4** *are satisfied.*

Proof Note that $\mathscr{H}$ is an H-field with $x' = 1$. The monotonicity of germs in $\mathscr{H}$ yields **HFC3**, whereas **HFC1**, **HFC2** and **HFC4** are immediate. $\square$

Proposition 4.15 *Let $\mathscr{H}$ be a Hardy field with composition and inversion. If there is no $f \in \mathscr{H}$ with $f > \exp^{[n]}$ for all $n \in \mathbb{N}$, then $\mathscr{H}$ satisfies* **HFC5**.

Proof By [38, Theorem 2], there is a Hardy field $\mathscr{H}^*$ containing $\mathscr{H}$ and which is closed under exp and log. We will partly work inside $\mathscr{H}^*$ so that we may compare our germs $f \in \mathscr{H}$ with elements of the form $\exp^{[n]}, n \in \mathbb{Z}$. Let $f, g \in \mathscr{H}^{>\mathbb{R}}$ and $\delta \in \mathscr{H} \cap \mathscr{o}(g)$ with $(f^{\dagger} \circ g)\delta \in \mathscr{o}$. Let $c \in \mathscr{C}^{<\infty}$ with $f \circ (g + \delta) - f \circ g = \delta f' \circ c$ as in Theorem 4.10. We will show that $f \circ (g + \delta) \in \Theta(f \circ g)$ by distinguishing two cases.

Case 1: $\exists p > 0, f \in \mathscr{O}((\log)^p)$ in $\mathscr{H}^*$. Then $f^{\dagger} \in \mathscr{O}\left(\frac{1}{\mathrm{id}\log}\right)$ so $f' \in \mathscr{O}\left(\frac{f}{\mathrm{id}\log}\right)$. Pick $r \in \mathbb{R}^{>}$ with $|f'| \leqslant r\frac{f}{\mathrm{id}\log}$. Consider a real number $s \in (0, 1)$. Recall that $\delta \in \mathscr{o}(g)$, so $- - s < \frac{c}{g} - 1 < s$. Lastly f' is the germ of a monotonic function. Combining all this, for sufficiently large $t > 1$, we have $(1 - s)g(t) < c(t)(1 + s)$, and

$$|f'(c(t))| \leqslant \frac{r \max(f(g(t) + \delta(t)), f(g(t)))}{(1 - s)g(t) \log(g(t))}.$$

Since $\delta \in \mathscr{o}(g)$ we get $\frac{|\delta(t)|}{g(t)} < \frac{1-s}{r}$ for sufficiently large $t > 1$. We deduce that

$$|\delta(t) f'(c(t))| \leqslant \frac{\max(f(g(t) + \delta(t)), f(g(t)))}{\log(g(t))}$$

for sufficiently large $t > 1$. Since $\log \circ g \notin \mathscr{O}$, we deduce that $\delta f' \circ c \in \mathscr{o}(f \circ (g + \delta))$ or $\delta f' \circ c \in \mathscr{o}(f \circ g)$. In particular $f \circ (g + \delta) \in \Theta(f \circ g)$.

Case 2: $\log^{\mathbb{N}} \subseteq \mathscr{O}(f)$ in $\mathscr{H}^*$. Our assumption on $\mathscr{H}$ implies that there are an $n \in \mathbb{N}$ and a $p \in \mathbb{N}$ with $\exp^{[n-2]} \in \mathscr{O}(f)$ and $f \in \mathscr{O}((\exp^{[n-1]})^p)$ in $\mathscr{H}^*$. We prove by induction on $k \leqslant n$ that for all $h \in \mathscr{H}^*$ with $h > \mathbb{R}$ and $p \in \mathbb{N}$ with $p > 0$, we have

$$(\exp^{[k-2]} \in \mathscr{O}(h) \wedge h \in \mathscr{O}((\exp^{[k-1]})^p) \Longrightarrow h \circ (g + \delta) \in \Theta(h \circ g)), \tag{17}$$

where $\exp^{[-2]} = \log^{[2]}$ and $\exp^{[-1]} = \log$. Note that for $k < n$, $p \in \mathbb{N}^{>}$ and $h \in (\mathscr{H}^*)^{>\mathbb{R}}$ with $\exp^{[k-2]} \in \mathscr{O}(h)$ and $h \in \mathscr{O}((\exp^{[k-1]})^p)$ we have $h \in \mathscr{O}(f)$ so $h^{\dagger} \in \mathscr{O}(f^{\dagger})$, so

$$(h^{\dagger} \circ g)\delta \in \mathscr{o}(1). \tag{18}$$

Thus if $k = 0$, then (17) follows from **Case 1**. Let $k < n$ such that (17) holds at k. Let $h \in (\mathscr{H}^*)^{>\mathbb{R}}$ and $p \in \mathbb{N}^{>}$ with $\exp^{[k-1]} \in \mathscr{O}(h)$ and $h \in \mathscr{O}((\exp^{[k]})^p)$. We again write $h \circ (g + \delta) - h \circ g = \delta h' \circ c$ where c lies strictly between g and $g + \delta$. It suffices to show that $\delta h' \circ c \in \mathcal{o}(h \circ g)$ or that $\delta h' \circ c \in \mathcal{o}(h \circ (g + \delta))$. Note that $h' \circ c \in \mathcal{O}(h' \circ g)$ or $h' \circ c \in \mathcal{O}(h' \circ (g + \delta))$ by monotonicity of h'. If $h' \circ c \in \mathcal{O}(h' \circ g)$, then (18) yields the result. So we may assume that $h' \circ c \in \mathcal{O}(h' \circ (g + \delta))$. We have $h \in \mathcal{O}((\exp^{[n]})^p)$, so $\log \circ h \in \mathcal{O}(\exp^{[n-1]})$, so $h^\dagger \in \mathcal{O}((\exp^{[n-1]})') \subseteq \mathcal{o}((\exp^{[n-1]})^2)$. The induction hypothesis at $n - 1$ for $h^\dagger$ yields $h^\dagger \circ (g + \delta) \in \Theta(h^\dagger \circ g)$, whence $(h^\dagger \circ (g + \delta))\delta \in \mathcal{o}$, whence $(h' \circ c)\delta \in \mathcal{o}(h \circ (g + \delta))$ as desired. By induction, the statement (17) holds for $k = n$, whence in particular $f \circ (g + \delta) \in \Theta(f \circ g)$.

We conclude with Theorem 4.13 that $\mathcal{H}$ satisfies **HFC5**. □

Corollary 4.16 *Let $\mathcal{H}$ be a Hardy field with composition and inversion. Assume that there is no germ $f \in \mathcal{H}$ with $f > \exp^{[n]}$ for all $n \in \mathbb{N}$. Then $\mathcal{H}$ is an H-field with composition and inversion.*

This result may extend to transexponential Hardy fields with composition and inversion, provided one has some control on the growth of elements of said fields. For instance, we believe it holds in Padgett's transexponential Hardy field with composition [32], provided it has inversion. In general, $\mathcal{H}^{>\mathbb{R}}$ should be contained in a single T–level as per [42] (see also [43]), for some o-minimal theory T.

Corollary 4.17 *Let $\mathcal{R}$ be an o-minimal expansion of the real ordered field in a first-order language $\mathcal{L}$. Assume that each $f \in \mathcal{R}_\infty$ lies below a germ $\exp^{[k]}, k \in \mathbb{N}$ in $(\mathcal{C}^{<\infty}, <)$. Let $\mathcal{R}^* = (\mathbb{R}^*, \ldots)$ be an elementary extension of $\mathcal{R}$. Consider the ordered field $\mathcal{R}^*_\infty$ with its canonical [12] derivation ∂, with the convex hull $\mathcal{O}^*$ of $\mathbb{R}^*$ as a valuation ring, and composition of germs. Then $\mathcal{R}^*_\infty$ is an H-field with composition and inversion.*

Proof The result holds, by Theorem 4.16, if $\mathcal{R}^* = \mathcal{R}$. The structure $(\mathcal{R}^*_\infty, \partial)$ is a differential field by [12, Chapter 7, Section 1.3]. Each element h of the valuation ring of $\mathcal{R}^*_\infty$ is the germ of a definable bounded monotonic function on $\mathbb{R}^*$, so by o-minimality of $\mathcal{R}^*$, it has a limit $c \in \mathbb{R}^*$. We have $h - c \in \mathcal{o}^*$ by definition, so **HF1** holds. If $h \in \mathcal{R}^*_\infty$ lies above $\mathbb{R}^*$, then by the monotonicity theorem h must be the germ of a strictly increasing function. We deduce with [12, Chapter 7, (2.5), Lemma 1] that $h' > 0$. So **HF2** holds and $\mathcal{R}^*_\infty$ is an H-field. Except for **HFC1** which refers to $\mathrm{Ker}(\partial) = \mathbb{R}^*$, all statements in Theorem 4.1 can be turned, after specialisation of the universally quantified variables, into sentences in $\mathcal{L}$. Since they hold for $\mathcal{R}_\infty$, they hold for $\mathcal{R}^*_\infty$. The existence of compositional inverses for elements in $\mathcal{G}_{\mathcal{R}^*}$ has already been established. This leaves the axiom **HFC1** to justify, but that follows immediately from the definition of $\mathbb{R}^*$. □

4.3 Conjugacy in H-fields with composition and inversion

We fix an H-field with composition and inversion $(K, +, \cdot, 0, 1, \mathcal{O}, <, \partial, \circ, x)$ over $\mathbb{R}$ and we write $\mathcal{G}$ for the group $K^{>\mathbb{R}}$ under composition.

Lemma 4.18 *Let $g = x + r_0 + \varepsilon$ where $r_0 \in \mathbb{R}$ and $\varepsilon \in \mathcal{o} \cap K^{>}$. Then $\mathscr{C}(g)$ is Archimedean and each $h \in \mathscr{C}(g)$ has the form $h = x + r + \delta$ for an $r \in \mathbb{R}$ and a $\delta \in \mathcal{o}$.*

Proof For $n \in \mathbb{Z}$, we claim that $(g^{[n]} - (x + nr_0)) \in \mathcal{o}$. Indeed this holds for $n = 0$. Given $n \in \mathbb{Z}$ such that $g^{[n]} = x + nr_0 + \varepsilon_n$ where $\varepsilon_n \in \mathcal{o}$, we have

$$g^{[n+1]} = x + nr_0 + \varepsilon_n + r_0 + \varepsilon \circ (x + nr_0 + \varepsilon_n) = x + (n+1)r_0 + \varepsilon_{n+1}$$

where $\varepsilon_{n+1} := \varepsilon \circ (x + nr_0 + \varepsilon_n) \in \mathcal{o}$ by **HFC1**. So we have the result for all $n \in \mathbb{N}$ by induction. Write $\varepsilon_{-1} := g^{[-1]} - x + r_0$. We have

$$x = g \circ (x - r_0 + \varepsilon_{-1}) = x + \varepsilon_{-1} + \varepsilon \circ (x - r_0 + \varepsilon_{-1})$$

where $\varepsilon \circ (x - r_0 + \varepsilon_{-1}) \in \mathcal{o}$ by **HFC1**. So we must have $\varepsilon_{-1} \in \mathcal{o}$, and we can use the same arguments as in the case $n \geqslant 0$, to show by induction that $(g^{[n]} - (x + nr_0)) \in \mathcal{o}$ for all $n \in -\mathbb{N}$.

Now let $h \in \mathscr{C}(g)^{>}$ and assume for contradiction that $\delta := h - x > \mathbb{R}$. Since $g \circ h = h \circ g$, we have $x + \delta + r_0 + \varepsilon \circ h = x + r_0 + \varepsilon + \delta \circ g$. So

$$\delta + \varepsilon \circ h = \varepsilon + \delta \circ g. \tag{19}$$

From $\varepsilon \in \mathcal{o}$ and $\varepsilon > 0$, we deduce by **HFC3** that $\varepsilon \circ h < \varepsilon$, whereas $\delta \circ g > \delta$. This contradicts (19). So $h = x + r + \iota$ for a certain $r \in \mathbb{R}$ and a certain $\iota \in \mathcal{o}$. Combining these two results, we deduce that $\mathscr{C}(g)$ is Archimedean. □

Lemma 4.19 *Let $f, g \in \mathcal{G}^{>}$ with $f > x + \mathbb{R}$, and assume that $g = x + 1 + \varepsilon$ for a certain $\varepsilon \in \mathcal{o}$ with $\varepsilon > 0$. Then $f \circ g > g \circ f$.*

Proof Recall that K is an H-field, so $f - x > \mathbb{R}$ entails that $(f - x)' > 0$, whence $f' > 1$. We distinguish three cases.

Assume that $f - x \in \mathcal{o}(x)$. So $f = x + \delta$ where $\delta > \mathbb{R}$. We have

$$\begin{aligned} f \circ g - g \circ f &= x + 1 + \varepsilon + \delta \circ (x + 1 + \varepsilon) - x - \delta - 1 - \varepsilon \circ (x + \delta) \\ &= (\delta \circ (x + 1 + \varepsilon) - \delta) + (\varepsilon - \varepsilon \circ (x + \delta)). \end{aligned}$$

Now $\delta \circ (x + 1 + \varepsilon) - \delta > 0$ because $\delta > \mathbb{R}$ and $x + 1 + \varepsilon > x$, and $\varepsilon - \varepsilon \circ (x + \delta) > 0$ because $\varepsilon \in K^{>} \cap \mathcal{o}$, $\varepsilon > 0$ and $x + \delta > x$. So $f \circ g > g \circ f$ in that case.

Assume now that $f \in \Theta(x)$. Then let $r \in \mathbb{R}$ with $f - rx \in \prec(x)$. So $r > 1$. Write $\delta := f - rx$, so $\delta \in \prec(x)$. This time, we have

$$f \circ g - g \circ f = (r-1) + (\delta \circ (x+1+\varepsilon) - \delta) + (r\varepsilon - \varepsilon \circ (x+\delta)).$$

As in the previous case, the term $\varepsilon - \varepsilon \circ (x+\delta)$ is strictly positive. We deduce since $r > 1$ that $r\varepsilon - \varepsilon \circ (x+\delta) > 0$. Since $r - 1 > 0$, it suffices to show that $\delta \circ (x+1+\varepsilon) - \delta \in \prec$. This is immediate if $\delta \in \preceq$. Indeed then we find by **HF1** an $r_0 \in \mathbb{R}$ and a $\iota \in \prec$ with $\delta = r_0 + \iota$. Thus

$$\delta \circ (x+1+\varepsilon) - \delta \in \Theta(\iota \circ (x+1+\varepsilon) - \iota),$$

whence $\delta \circ (x+1+\varepsilon) - \delta \in \prec$ by **HFC1**.

Assume that $\delta \notin \preceq$. Recall that $\delta \in \prec(x)$, so in view of (13), we have $\delta' \in \prec$. Therefore $\delta'(1+\varepsilon) \in \prec$. By **HFC5**, we have

$$\delta \circ (x+1+\varepsilon) - \delta - \delta'(1+\varepsilon) \in \prec(\delta'(1+\varepsilon)).$$

Since $\delta'(1+\varepsilon) \in \prec$, we must have $\delta \circ (x+1+\varepsilon) - \delta \in \prec$ as claimed.

We finally treat the remaining case when $f/x > \mathbb{R}$. We have $f > x\mathbb{R}^{>}$, so $f^{\mathrm{inv}} < \mathbb{R}^{>}x$. Since $f^{\mathrm{inv}} > \mathbb{R}$, we deduce with **HF2** that $0 < (f^{\mathrm{inv}})' < \mathbb{R}^{>}$, ie $(f^{\mathrm{inv}})' \in \prec$. It suffices to show that $g^{\mathrm{inv}} \circ f^{\mathrm{inv}} < f^{\mathrm{inv}} \circ g^{\mathrm{inv}}$. Recall as in the proof of Lemma 4.18 that $g^{\mathrm{inv}} = x - 1 - \delta$ for a certain $\delta \in \prec \cap K^{>}$. We have $(f^{\mathrm{inv}})^{\dagger} = \frac{(f^{\mathrm{inv}})'}{f^{\mathrm{inv}}} \in \prec$. Since $(f^{\mathrm{inv}})^{\dagger} 1 \in \prec$, the axiom **HFC5** gives

$$f^{\mathrm{inv}} \circ (x - 1 - \delta) \in f^{\mathrm{inv}} - (f^{\mathrm{inv}})'(1+\delta) + \prec((f^{\mathrm{inv}})').$$

Therefore $f^{\mathrm{inv}} \circ g^{\mathrm{inv}} - f^{\mathrm{inv}} \in \prec$. We have

$$g^{\mathrm{inv}} \circ f^{\mathrm{inv}} - f^{\mathrm{inv}} = (x - 1 - \delta) \circ f^{\mathrm{inv}} - f^{\mathrm{inv}} = -1 - \delta \circ f^{\mathrm{inv}} \in -1 + \prec.$$

Thus $g^{\mathrm{inv}} \circ f^{\mathrm{inv}} - f^{\mathrm{inv}} < f^{\mathrm{inv}} \circ g^{\mathrm{inv}} - f^{\mathrm{inv}}$, so $g^{\mathrm{inv}} \circ f^{\mathrm{inv}} < f^{\mathrm{inv}} \circ g^{\mathrm{inv}}$. □

We next need to find approximate primitives of elements in K. These are large enough that this does not require any further assumption on K (such as having asymptotic integration, see [2, page 8]).

Lemma 4.20 *Given $\delta \in \preceq(x^{-2})$, there is an $h \in K$ with $h' - \delta^{-1} \in \prec(\delta^{-1}) \cap K^{>}$.*

Proof In view of [38, Theorem 1], it suffices to show that $x^{-2} \in \prec(f^{\dagger})$ for all $f \in \prec \setminus \{0\}$. Let $f \in \prec \setminus \{0\}$. By (12), we have $g' \in \prec(f^{\dagger})$ for all $g \in \prec$. In particular $(x^{-1})' = -x^{-2} \in \prec(f)$, hence the result. □

Lemma 4.21 *Let $g \in \mathcal{G}^{>}$ be of the form $g = x + \delta$ where $\delta \in K^{>} \cap \mathcal{O}(x^{-2})$. There are an $h \in \mathcal{G}$ and an $\varepsilon \in \scriptstyle{\mathcal{O}}$ with $\varepsilon > 0$ and $h \circ g \circ h^{\mathrm{inv}} = x + 1 + \varepsilon$.*

Proof By Theorem 4.20, the condition on δ implies that there is an $h \in K$ such that the germ $\iota := h' - \delta^{-1}$ satisfies $\iota \in \scriptstyle{\mathcal{O}}(\delta^{-1})$ and $\iota > 0$. Since $\delta \in K^{>} \cap \scriptstyle{\mathcal{O}}$, the element δ^{-1} is positive infinite. Note that $\mathcal{O}' = \scriptstyle{\mathcal{O}}' \subseteq \scriptstyle{\mathcal{O}}$ by (14), while $f' < 0$ for all negative infinite elements by **HF1**. So $h \in \mathcal{G}$. We have $\delta \circ h^{\mathrm{inv}} \in \scriptstyle{\mathcal{O}}(h)$ because $\delta \in \mathcal{O}$ whereas $h \notin \mathcal{O}$. Finally, we have

$$\delta h^{\dagger} \in \Theta\left(\frac{\delta}{h\delta}\right) \quad \text{and} \quad \frac{1}{h} \in \scriptstyle{\mathcal{O}},$$

so $\delta h^{\dagger} \in \scriptstyle{\mathcal{O}}$. Consider by **HFC5** the Taylor approximation

$$\begin{aligned} h \circ g \circ h^{\mathrm{inv}} &= h \circ (h^{\mathrm{inv}} + \delta \circ h^{\mathrm{inv}}) \\ &= x + (h' \circ h^{\mathrm{inv}})(\delta \circ h^{\mathrm{inv}}) + \frac{1}{2}(h'' \circ h^{\mathrm{inv}})(\delta \circ h^{\mathrm{inv}})^2 + \delta_1 \end{aligned}$$

where $\delta_1 \in \scriptstyle{\mathcal{O}}((h'' \circ h^{\mathrm{inv}})(\delta \circ h^{\mathrm{inv}})^2)$. Note that

$$(h' \circ h^{\mathrm{inv}})(\delta \circ h^{\mathrm{inv}}) = (1 + \iota\delta) \circ h^{\mathrm{inv}} = 1 + (\iota\delta) \circ h^{\mathrm{inv}}$$

where $(\iota\delta) \circ h^{\mathrm{inv}}$ is positive by **HFC1**. We have $h'' = (\delta^{-1})' + \iota'$. Now L'Hospital's rule (13) entails that the sign of h'' is that of $(\delta^{-1})'$, which is positive because $\delta^{-1} > \mathbb{R}$. So $h''\delta^2 > 0$, so $\frac{1}{2}(h'' \circ h^{\mathrm{inv}})(\delta \circ h^{\mathrm{inv}})^2 > 0$ whence

$$\varepsilon := (\iota\delta) \circ h^{\mathrm{inv}} + \frac{1}{2}(h'' \circ h^{\mathrm{inv}})(\delta \circ h^{\mathrm{inv}})^2 + \delta_1 > 0.$$

We have $h \circ g \circ h^{\mathrm{inv}} = x + 1 + \varepsilon$ as desired. □

4.4 Ordered groups in H-fields with composition and inversion

We now prove Theorem 4.6. Let $(K, +, \cdot, 0, 1, \partial, \mathcal{O}, <, \circ, x)$, $\mathcal{G}_0$ and $\mathcal{G}_1$ be as in the statement of Theorem 4.6. Consider the projections $\pi_0 : K^{>\mathbb{R}} \longrightarrow \mathcal{G}_0$ and $\pi_1 : K^{>\mathbb{R}} \longrightarrow \mathcal{G}_1$ with $\pi_0\pi_1 = \mathrm{Id}_{K^{>\mathbb{R}}}$.

Lemma 4.22 *For all $g \in \mathcal{G}_0^{>}$, there are a $\varphi \in \mathcal{G}_1$ and an $\varepsilon \in K^{>} \cap \scriptstyle{\mathcal{O}}$ with*

$$\varphi \circ g \circ \varphi^{\mathrm{inv}} = x + 1 + \varepsilon.$$

Proof Let $\varphi \in \mathcal{G}_1$ such that $\varphi > x^3 + \mathbb{R}$ and $g \leqslant \varphi \circ (\varphi^{\mathrm{inv}} + 1)$. Thus $\varphi^{\mathrm{inv}} \circ g \circ \varphi \leqslant x + 1$. We have so $x \leqslant \varphi^{[-3]} \circ g \circ \varphi^{[3]}$ because g is positive in the ordered group $\mathcal{G}_0$. So

$x \leqslant \varphi^{[-2]} \circ (\varphi^{[2]} + 1)$. Note that $\varphi > x + \mathbb{R}$, so $\varphi' > 1$ by **HF2**. We have $(\varphi^{\mathrm{inv}})^{\dagger} \circ \varphi = \frac{1}{\varphi' x} \in o(1)$. **HFC5** for $(\varphi^{\mathrm{inv}}, \varphi, 1)$ gives $\varphi^{\mathrm{inv}} \circ (\varphi + 1) - x - (\varphi^{\mathrm{inv}})' \circ \varphi \in o((\varphi^{\mathrm{inv}})' \circ \varphi)$, so

$$\varphi^{\mathrm{inv}} \circ (\varphi + 1) - x \in \mathcal{O}\left(\frac{1}{\varphi'}\right).$$

But $\varphi > x^3 + C$ so $\varphi' > 3x^2$, so $\delta := \varphi^{\mathrm{inv}} \circ (\varphi + 1) - x$ lies in $\mathcal{O}(x^{-2})$. We deduce that Theorem 4.21 applies and yields the result. □

Lemma 4.23 *For $g \in \mathcal{G}_0^{\neq}$, we have $\mathcal{C}(g) \subseteq \mathcal{G}_0$.*

Proof We may assume that $g > x$. Let $\varphi \in \mathcal{G}_1$ be given by Lemma 4.22 with $\varphi^{\mathrm{inv}} \circ g \circ \varphi = x + 1 + \varepsilon$ for an $\varepsilon \in K^{>} \cap o$. We have $\mathcal{C}(g) = \varphi \circ \mathcal{C}(x + 1 + \varepsilon) \circ \varphi^{\mathrm{inv}}$, so it suffices to show that $\mathcal{C}(x + 1 + \varepsilon) \subseteq \mathcal{G}_0$. This follows from Theorem 4.18 and the fact that $\mathcal{G}_0$ is a convex subgroup of $K^{>\mathbb{R}}$. □

We will use the identity $\mathcal{C}(g) = \mathcal{C}(g) \cap \mathcal{G}_0$ for $g \in \mathcal{G}_0$ without mention.

Corollary 4.24 *The subgroup $\mathcal{G}_0 \subseteq K^{>\mathbb{R}}$ is $\preccurlyeq$–initial in $K^{>\mathbb{R}}$.*

Proposition 4.25 *The group $K^{>\mathbb{R}}$ has Archimedean centralisers.*

Proof Let $g \in K^{>\mathbb{R}}$ with $g > x$. If $g \in \mathcal{G}_0$, then by Lemma 4.22 the ordered group $\mathcal{C}(g)$ is isomorphic to $\mathcal{C}(x + 1 + \varepsilon)$ for an $\varepsilon \in K^{>} \cap o$, whence $\mathcal{C}(g)$ is Archimedean by Theorem 4.18.

If $g \notin \mathcal{G}_0$, then we must have $g > \mathcal{G}_0$ by convexity. For $f, h \in K^{>\mathbb{R}}$, we have $[f, h] = 1 \Longrightarrow [\pi_1(f), \pi_1(h)] = \pi_1(1) = 1$, so the morphism $\pi_1 \upharpoonright \mathcal{C}(g) : \mathcal{C}(g) \longrightarrow \mathcal{G}_1$ ranges in $\mathcal{C}(\pi_1(g)) \cap \mathcal{G}_1$. It is nondecreasing by (9). For $h \in \operatorname{Ker}(\pi_1) \cap \mathcal{C}(g) = \mathcal{G}_0 \cap \mathcal{C}(g)$, since $g \notin \mathcal{G}_0$, we cannot have $h \in \mathcal{G}_0^{\neq}$ by Theorem 4.23. Therefore $\pi_1 \upharpoonright \mathcal{C}(g)$ is an embedding of ordered groups $\mathcal{C}(g) \longrightarrow \mathcal{C}(\pi_1(g)) \cap \mathcal{G}_1$. We deduce since its codomain is Archimedean that $\mathcal{C}(g)$ is Archimedean. □

Corollary 4.26 *The axiom* **GOG1** *holds in $\mathcal{G}_0$ and in $K^{>\mathbb{R}}$.*

Proof For $K^{>\mathbb{R}}$ this follows from Propositions 4.25 and 2.1. For $\mathcal{G}_0$, we know by Propositions 4.25 and Theorem 4.23 that it has Archimedean centralisers. We conclude with Proposition 2.1. □

Lemma 4.27 *The axiom* **GOG2** *holds in $\mathcal{G}_0$ and in $K^{>\mathbb{R}}$.*

Proof Let $f, g \in K^{>\mathbb{R}}$ with $f, g \geqslant x$. Suppose first that $g \in \mathscr{G}_0$ and $f > \mathscr{C}(g)$. We may assume by Theorem 4.22 that $g = x + 1 + \varepsilon$ for an $\varepsilon \in K^{>} \cap \mathcal{o}$. We must have $f > x + \mathbb{R}$ by Theorem 4.18, whence $f \circ g > g \circ f$ by Theorem 4.19. Applying this for $f \in \mathscr{G}_0$, we see that **GOG2** holds in $\mathscr{G}_0$.

Suppose now that $f > \mathscr{C}(g)$. If $g \in \mathscr{G}_0$, then the arguments above apply and yield $f \circ g > g \circ f$. If not, we have $\pi_1(g) > x$ since $\mathscr{G}_0$ is a convex subgroup of $K^{>\mathbb{R}}$. Recall that $\mathscr{C}(\pi_1(g)) \cap \mathscr{G}_1$ is Archimedean, so $\pi_1(g)^{[\mathbb{N}]}$ is cofinal in it. We have $\pi_1(f) > \pi_1(\mathscr{C}(g)) \supseteq \pi_1(g^{[\mathbb{N}]}) = \pi_1(g)^{[\mathbb{N}]}$, so $\pi_1(f) > \mathscr{C}(\pi_1(g)) \cap \mathscr{G}_1$. Thus **GOG2** in $\mathscr{G}_1$ yields

$$\pi_1(f \circ g) = \pi_1(f) \circ \pi_1(g) > \pi_1(g) \circ \pi_1(f) = \pi_1(g \circ f).$$

By (9) and by convexity of $\mathscr{G}_0$, we have $f \circ g > g \circ f$. So **GOG2** holds. □

Lemma 4.28 *The axiom* **GOG3** *holds in* $\mathscr{G}_0$ *and in* $K^{>\mathbb{R}}$.

Proof Let $g \in K^{>\mathbb{R}}$ with $g > x$. Suppose first that $g \in \mathscr{G}_0$. Let $\varphi \in \mathscr{G}_1$ with $\varphi^{\mathrm{inv}} \circ g \circ \varphi = x + 1 + \varepsilon$ for some $\varepsilon \in K^{>} \cap \mathcal{o}$. By Theorem 2.17 and Theorem 4.18, the the element $x + 1$ is scaling in $K^{>\mathbb{R}}$ with $x + 1 \asymp \varphi^{\mathrm{inv}} \circ g \circ \varphi$ in $K^{>\mathbb{R}}$. The conjugation by φ is an automorphism of $(\mathscr{G}_0, \circ, x, <)$, so the element $\mathfrak{s} := \varphi \circ (\varphi^{\mathrm{inv}} + 1) \in \mathscr{G}_0$ is scaling in $\mathscr{G}_0$ with $\mathfrak{s} \asymp g$. Thus $\mathscr{G}_0$ has scaling elements. In view of Theorem 4.23, we also obtain that $\mathfrak{s}$ is also scaling in $K^{>\mathbb{R}}$ with $\mathfrak{s} \asymp g$ in $K^{>\mathbb{R}}$.

Now suppose that $g \notin \mathscr{G}_0$, so $g > \mathscr{G}_0$ by convexity. Let $\mathfrak{t} \in \mathscr{G}_1$ be scaling in $\mathscr{G}_1$ with $\mathfrak{t} \asymp \pi_1(g)$ in $\mathscr{G}_1$. Since $\mathscr{C}(\pi_1(g)) \cap \mathscr{G}_1$ is Archimedean, we have $\pi_1(g)^{[-n]} \leqslant \mathfrak{t} \leqslant \pi_1(g)^{[n]}$ for some $n \in \mathbb{N}$, so $g^{[-n-1]} \leqslant \mathfrak{t} \leqslant g^{[n+1]}$, whence $\mathfrak{t} \asymp g$ in $K^{>\mathbb{R}}$. We claim that $\mathfrak{t}$ is scaling in $K^{>\mathbb{R}}$. Indeed let $f \in K^{>\mathbb{R}}$ with $f \asymp g$. By Theorem 4.25, we have $g^{[-n]} \leqslant f \leqslant g^{[n]}$ for some $n \in \mathbb{N}$, so $\pi_1(g^{[-n]}) \leqslant \pi_1(f) \leqslant \pi_1(g^{[n]})$, whence $\pi_1(f) \asymp \mathfrak{t}$ in $\mathscr{G}_1$. Let $\mathfrak{u} \in \mathscr{C}(\mathfrak{t}) \cap \mathscr{G}_1$ with $\pi_1(f)\mathfrak{u}^{-1} \prec \pi_1(f)$ in $\mathscr{G}_1$. Since $\mathscr{G}_1$ has Archimedean centralisers, this means that $\pi_1((f\mathfrak{u}^{-1})^{[\mathbb{Z}]}) = (\pi_1(f)\mathfrak{u}^{-1})^{[\mathbb{Z}]} < \pi_1(f)$, whence $(f\mathfrak{u}^{-1})^{[\mathbb{Z}]} < f$. We deduce with Theorem 4.25 that $f \sim \mathfrak{u}$ in $K^{>\mathbb{R}}$. Thus $\mathfrak{t}$ is scaling in $K^{>\mathbb{R}}$. Therefore **GOG3** holds in $K^{>\mathbb{R}}$. □

This concludes the proof of Theorem 4.6.

4.5 Application in the polynomially bounded case

Let $\mathscr{R}$ be an o-minimal expansion of $(\mathbb{R}, +, \cdot, <)$. We recall a fundamental dichotomy for the asymptotic growth of germs in $\mathscr{G}_{\mathscr{R}}$:

Miller's dichotomy [28] If there is an $f \in \mathscr{R}_\infty$ with $f > \mathrm{id}^n$ for all $n \in \mathbb{N}$, then the exponential function is definable in $\mathscr{R}$.

If exp is not definable, then $\mathscr{R}$ is said *polynomially bounded*. Let us first work on that smaller side of the dichotomy, that is, suppose that $\mathscr{R}$ is polynomially bounded. Let E denote the set of real numbers e such that the germ id^e of the e–power function is in $\mathscr{R}_\infty$. It is easy to see that E is a subfield of $\mathbb{R}$.

By [28, Proposition], for each $f \in \mathscr{R}_\infty$, there is a unique $(e_f, c_f) \in E \times \mathbb{R}$ such that $f - c_f \,\mathrm{id}^{e_f} \in \prec\!\!\!\!\circ(f)$. If $f > \mathbb{R}$, then we must have $e_f > 0$ and $c_f > 0$. Note that $\mathbb{R}^>$ is an ordered vector space over E, and thus we have a growth order group $\mathrm{Aff}_E(\mathbb{R}^>)$ as in Theorem 3.4. We set $\mathcal{G}_1 := \mathbb{R}^> \mathrm{id}^{E^>}$. Note that the function

$$\begin{array}{rcl} (e_., c_.) : \mathcal{G}_{\mathscr{R}} & \longrightarrow & \mathrm{Aff}_E(\mathbb{R}^>) \\ f & \longmapsto & (e_f, c_f) \end{array}$$

is a homomorphism of ordered groups which restricts to an isomorphism

$$\mathcal{G}_1 \longrightarrow \mathrm{Aff}_E(\mathbb{R}^>).$$

Therefore $\mathcal{G}_1 \simeq \mathrm{Aff}_E(\mathbb{R}^>)$ is a growth order group. Let $\mathcal{G}_0$ denote the kernel of $(e_., c_.)$. So $\mathcal{G}_0$ is a normal subgroup of $\mathcal{G}_{\mathscr{R}}$ and $\mathcal{G}_1$ is a complement of $\mathcal{G}_0$ in $\mathcal{G}_{\mathscr{R}}$. Here $\mathcal{G}_0$ corresponds to germs that are tangent to the identity, whereas $\mathcal{G}_1$ is a group of non-monic monomials.

Proposition 4.29 *The ordered pair* $(\mathcal{G}_0, \mathcal{G}_1)$ *satisfies* $(\star)$ *for* $\mathcal{G}_{\mathscr{R}}$.

Proof We have $\mathcal{G}_0 = \{g \in \mathcal{G}_{\mathscr{R}} : g - x \in \prec\!\!\!\!\circ(\mathrm{id})\}$, so $\mathcal{G}_0$ is a convex subgroup of $\mathcal{G}_{\mathscr{R}}$ which contains $\mathrm{id} + 1$. For $c \in \mathbb{R}^> \setminus \{1\}$, the centraliser of $(1, c)$ in $\mathrm{Aff}_E(\mathbb{R}^>)$ is $\{1\} \times \mathbb{R}^> \simeq \mathbb{R}^>$. In Theorem 3.4, we saw that given $e \in E^>$ with $e \neq 1$ and $c \in \mathbb{R}^>$, for all $q \in E^>$, there is a unique $c_0 \in \mathbb{R}^>$ such that (q, c_0) and (e, c) commute. Thus the projection on the first variable is an isomorphism between $\mathscr{C}((e, c))$ and $E^>$. Note that $E^>$ embeds into the Archimedean ordered group $(\mathbb{R}^>, \cdot, 1, <) \simeq (\mathbb{R}, +, 0, <)$, it is Archimedean. Therefore $\mathcal{G}_1$ has Archimedean centralisers.

It remains to show that $L := \{f \circ (f^{\mathrm{inv}} + 1) : f \in \mathcal{G}_1\}$ is cofinal in $\mathcal{G}_0$. Let $g = x + \delta \in \mathcal{G}_0$, so $\delta \in \prec\!\!\!\!\circ(\mathrm{id})$. We have $\delta - cx^e \in \prec\!\!\!\!\circ(\delta)$ for a certain $(e, c) \in E \times \mathbb{R}$. The condition $\delta \in \prec\!\!\!\!\circ(\mathrm{id})$ implies that $e < 1$, so we find an $n \in \mathbb{N}$ with $\frac{2^n - 1}{2^n} > e$. Note that

$$\mathrm{id}^{2^n} \circ (\mathrm{id}^{2^{-n}} + 1) \in \mathrm{id} + 2^n \,\mathrm{id}^{\frac{2^n-1}{2^n}} + \prec\!\!\!\!\circ(\mathrm{id}^{\frac{2^n-1}{2^n}}).$$

Therefore $\mathrm{id}^{2^n} \circ (\mathrm{id}^{2^{-n}} + 1) > g$. This implies that L is cofinal in $\mathcal{G}_0$. □

As $\mathcal{R}$ is polynomially bounded, Theorem 4.16 applies and entails that $\mathcal{R}_\infty$ is an H-field with composition and inversion. Theorem 4.6 gives:

Corollary 4.30 *Let $\mathcal{R}$ be a polynomially bounded o-minimal expansion of the real ordered field. Then $\mathcal{G}_{\mathcal{R}}$ is a growth order group with Archimedean centralisers.*

4.6 Applications in the exponential case

In order to deal with the exponential case, we introduce a notion of H-field with an exponential function. We will also give additional applications of Theorem 4.6.

Definition 4.31 *An **exponential H-field** is an H-field K over $\mathbb{R}$ together with an isomorphism $\log : (K^>, \cdot, 1, <) \longrightarrow (K, +, 0, <)$, whose reciprocal is denoted $\exp$, such that*

$$\log(1 + \scriptstyle{\mathcal{O}}) = \scriptstyle{\mathcal{O}} \quad \textit{and} \tag{20}$$

$$\forall a \in K^>, a^\dagger = (\log a)'. \tag{21}$$

Thus $(K, +, \cdot, 0, 1, <, \exp)$ is an ordered exponential field as per [24]. We fix an exponential H-field K. Consider a Hardy field with composition $\mathcal{H}$ containing log and a morphism of ordered valued differential fields $\Phi : \mathcal{H} \longrightarrow K$. For all $f \in \mathcal{H}^>$, we have $\Phi(f) > 0$ and

$$(\log \Phi(f))' = \frac{\Phi(f)'}{\Phi(f)} = \frac{\Phi(f')}{\Phi(f)} = \Phi(\frac{f'}{f}) = \Phi((\log \circ f)') = \Phi(\log \circ f)'$$

by (21). So $\log \Phi(f) - \Phi(\log \circ f) \in \mathbb{R}$. For all $a \in K^{>\mathbb{R}}$ and $\delta \in \scriptstyle{\mathcal{O}}(a)$, we have $\log(a+\delta) - \log(s) \in \scriptstyle{\mathcal{O}}$. Indeed $\log(a+\delta) = \log(a(1+\delta a^{-1})) = \log(a) + \log(1+\delta a^{-1})$ where $\log(1 + \delta a^{-1}) \in \scriptstyle{\mathcal{O}}$ by (20). An induction gives

$$\log^{[k]} \Phi(f) - \Phi(\log^{[k]} \circ f) \in \scriptstyle{\mathcal{O}} \tag{22}$$

for all $f \in \mathcal{H}^{>\mathbb{R}}$ and $k > 1$.

Proposition 4.32 *Let $\mathcal{H}$ be a Hardy field with composition and inversion containing $\exp$ and let $\Phi : \mathcal{H} \longrightarrow K$ be embedding of ordered valued differential fields. Set $x := \Phi(\mathrm{id})$ and suppose that for all $a \in K^{>\mathbb{R}}$, there is an $l \in \mathbb{Z}$ such that for all sufficiently large $k \in \mathbb{N}$, we have*

$$\log^{[k]}(a) - \log^{[k-l]}(x) \in \scriptstyle{\mathcal{O}}. \tag{23}$$

Then $\mathcal{H}^{>\mathbb{R}}$ is a growth order group with Archimedean centralisers.

Proof We will write $\mathcal{o}_K := \mathcal{o}(1) \subseteq K$ and $\mathcal{o}_{\mathscr{H}} := \mathcal{o}(1) \subseteq \mathscr{H}$. Consider the subgroup $\mathscr{G}_1 := \exp^{[\mathbb{Z}]}$ of $\mathscr{H}^{>\mathbb{R}}$. This is a growth order group with Archimedean centralisers as it is itself Archimedean. Let $\mathscr{G}_0$ denote the subset of $\mathscr{H}^{>\mathbb{R}}$ of elements g with $g^{[\mathbb{Z}]} < \exp$. This is a convex subgroup of $\mathscr{H}^{>\mathbb{R}}$ containing $\mathrm{id} + \mathbb{R}$. We claim that $(\mathscr{G}_0, \mathscr{G}_1)$ satisfies $(\star)$. We have $\mathscr{G}_1 \cap \mathscr{G}_0 = \{\mathrm{id}\}$ by definition. Let us show that $\mathscr{H}^{>\mathbb{R}} = \mathscr{G}_0\mathscr{G}_1$.

Let $f \in \mathscr{H}$ with $f \geqslant \mathrm{id}$. By (22) and (23), we find an $l \in \mathbb{Z}$ such that for large enough $k > 1$, the element $\log^{[k]}(\Phi(f)) - \log^{[k]}(\exp^{[l]}(x))$ lies in $\mathcal{o}_K$. We claim that $g := f \circ \log^{[l]} \in \mathscr{G}_0$. By (22), given $k > 1$ large enough, we have $\Phi(\log^{[k]} \circ f) - \Phi(\log^{[k-l]}(\mathrm{id})) \in \mathcal{o}_K$, whence $\log^{[k]} \circ f - \log^{[k-l]}(\mathrm{id}) \in \mathcal{o}_{\mathscr{H}}$. Thus $\log^{[k]} \circ f \circ \exp^{[k]} - \exp^{[l]}$ and $\log^{[k]} \circ g \circ \exp^{[k]} - \mathrm{id}$ lie in $\mathcal{o}_{\mathscr{H}}$. But then $\log^{[k]} \circ g \circ \exp^{[k]} \leqslant \mathrm{id} + 1$ so $g^{[n]} < \exp^{[k]} \circ (\mathrm{id} + n) \circ \log^{[k]} \leqslant \exp^{[k]} \circ \exp \circ \log^{[k]} = \exp$ for all $n \in \mathbb{N}$, ie $g \in \mathscr{G}_0$.

For $h \in \mathscr{G}_1$, $g \in \mathscr{G}_0$ and $n \in \mathbb{N}$, we have $(h \circ g \circ h^{\mathrm{inv}})^{[n]} = h \circ g^{[n]} \circ h^{\mathrm{inv}} < h \circ \exp \circ h^{\mathrm{inv}} = \exp$. So $h \circ \mathscr{G}_0 \circ h^{\mathrm{inv}} \subseteq \mathscr{G}_0$. It follows since $\mathscr{H}^{>\mathbb{R}} = \mathscr{G}_0\mathscr{G}_1$ that $\mathscr{G}_0$ is a normal subgroup of $\mathscr{H}^{>\mathbb{R}}$.

Finally, assume for contradiction that $g > \exp^{[k]} \circ (\log^{[k]} + 1)$ for some $g \in \mathscr{G}_0$, for all $k > 1$. By (22), for each $k > 1$, we have a $\delta_k \in \mathcal{o}_K$ with $\log^{[k]}(\Phi(g)) + \delta_k > \log^{[k]}(x) + 1$. In particular $\log^{[k]} \Phi(g) > \log^{[k]}(x) + \frac{1}{2}$, whence $\Phi(g) > \exp^{[k]}(\log^{[k]}(x) + \frac{1}{2})$, for all $k > 1$. Let $\ell \in \mathbb{Z}$ and $k_0 > 1$ with $\log^{[k_0]}(\Phi(g)) - \log^{[k_0 - \ell]}(x) \in \mathcal{o}_K$. We have $\ell > 0$ since $\log^{[k_0]}(x) + \frac{1}{4} < \log^{[k_0 - \ell]}(x)$. Now (22) gives $\Phi(\log^{[k_0]}(g) - \log^{[k_0 - \ell]}) \in \mathcal{o}_K$, so $\log^{[k_0]}(g) - \log^{[k_0 - \ell]} \in \mathcal{o}_{\mathscr{H}}$. In particular $\log^{[k_0]}(g) - \log^{[k_0 - \ell]} \geqslant -1$, thus

$$\begin{aligned} g^{[2]} &= \exp^{[k_0]}(\log^{[k_0]}(g)) \circ \exp^{[k_0]}(\log^{[k_0]}(g)) \\ &\geqslant \exp^{[k_0]} \circ (\mathrm{id} - 1) \circ \log^{[k_0 - \ell]} \circ \exp^{[k_0]} \circ (\mathrm{id} - 1) \circ \log^{[k_0 - \ell]} \\ &\geqslant \exp^{[k_0]} \circ ((\mathrm{id} - 1) \circ \exp^{[\ell]} \circ (\mathrm{id} - 1) \circ \exp^{[\ell]}) \circ \log^{[k_0]}. \end{aligned}$$

We have $(\mathrm{id} - 1) \circ \exp^{[\ell]} \circ (\mathrm{id} - 1) \circ \exp^{[\ell]}) = h \circ \exp^{[2\ell]}$, where

$$h := (\mathrm{id} - 1) \circ (\exp^{[\ell]} \circ (\mathrm{id} - 1) \circ \log^{[\ell]}).$$

Now $h \in \mathscr{G}_0$ by our previous arguments, so $h \geqslant \log$, so

$$g^{[2]} \geqslant \exp^{[k_0]} \circ h \circ \exp^{[2\ell]} \circ \log^{[k_0]} \geqslant \exp^{[2\ell - 1]}.$$

This contradicts the assumption that $g \in \mathscr{G}_0$, and thus concludes out proof that $\{f \circ (f^{\mathrm{inv}} + 1) : f \in \mathscr{G}_1\}$ is cofinal in $\mathscr{G}_0$. So $(\star)$ holds. We conclude with Theorem 4.6. □

Corollary 4.33 *Let $\mathscr{P}$ be the Pfaffian closure of the real ordered field [41]. Then $\mathscr{G}_{\mathscr{P}}$ is a growth order group with Archimedean centralisers.*

Proof The field $\mathbb{T}_{\mathrm{LE}}$ of logarithmic-exponential transseries is an exponential H-field (see [13, 3]). The property (23) holds [27, Claim, page 248] in $\mathbb{T}_{\mathrm{LE}}$. We have an embedding of ordered valued differential fields [4, Corollary 7.3.4] of $\mathscr{P}_\infty$ into $\mathbb{T}_{\mathrm{LE}}$. So Theorem 4.32 applies. □

Let us complete our proof of Theorem 2. Let $\mathscr{R}$ be a levelled expansion of the real ordered field that is not polynomially bounded. We have $\exp, \log \in \mathscr{G}_{\mathscr{R}}$ by Miller's dichotomy. This yields an isomorphism of ordered groups

$$\begin{aligned} \log : \mathscr{R}_\infty^{>} &\longrightarrow \mathscr{R}_\infty \\ f &\longmapsto \log \circ f \end{aligned}$$

and $(\mathscr{R}_\infty, \log)$ is an exponential H-field (see [24, Section 6.2]). Since $\mathscr{R}$ is levelled and in view of (22), the condition (23) holds. Theorem 4.32 gives:

Corollary 4.34 *Let $\mathscr{R}$ be a levelled o-minimal expansion of the real ordered field that is not polynomially bounded. Then $\mathscr{G}_{\mathscr{R}}$ is a growth order group with Archimedean centralisers.*

Remark 5 Any reduct of a levelled o-minimal expansion of the real ordered field that defines the sum and product is clearly a levelled o-minimal expansion of the real ordered field, therefore it also induces a growth order group.

Corollaries 4.30 and 4.34 imply Theorem 2. By [23, Theorem 1], we have:

Corollary 4.35 *Let $\mathscr{R}$ be an o-minimal expansion of the real ordered field by the exponential and a generalised quasianalytic class* [35] *containing the restricted analytic* exp *and* log. *Then $\mathscr{G}_{\mathscr{R}}$ is a growth order group with Archimedean centralisers.*

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