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Multi-level nonstandard analysis and the axiom of choice

KAREL HRBACEK

Abstract: Model-theoretic frameworks for Nonstandard Analysis depend on the existence of nonprincipal ultrafilters, a strong form of the Axiom of Choice (**AC**). Hrbacek and Katz in Annals of Pure and Applied Logic 72 (2021) formulate axiomatic nonstandard set theories **SPOT** and **SCOT** that are conservative extensions of respectively **ZF** and **ZF** + **ADC** (the Axiom of Dependent Choice), and in which a significant part of Nonstandard Analysis can be developed. The present paper extends these theories to theories with many levels of standardness, called respectively **SPOTS** and **SCOTS**. It shows that Jin's recent nonstandard proof of Szemerédi's Theorem can be carried out in **SPOTS**, which is conservative over **ZF**+**ACC** (the Axiom of Countable Choice). The theory **SCOTS** is a conservative extension of **ZF** + **ADC**.

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1 Introduction

Nonstandard Analysis is sometimes criticized for its implicit dependence on the Axiom of Choice (**AC**) (see eg Connes [5][1]). Indeed, model-theoretic frameworks based on hyperreals require the existence of nonprincipal ultrafilters over $\mathbb{N}$, a strong form of **AC**:

> If $*$ is the mapping that assigns to each $X \subseteq \mathbb{N}$ its nonstandard extension ${}^*X$, and if $\nu \in {}^*\mathbb{N} \setminus \mathbb{N}$ is an unlimited integer, then the set $U = \{X \subseteq \mathbb{N} \mid \nu \in {}^*X\}$ is a nonprincipal ultrafilter over $\mathbb{N}$.

The common axiomatic/syntactic frameworks for nonstandard methods (see Kanovei and Reeken [23]), such as **IST** or **HST**, include **ZFC** among their axioms. The dependence on **AC** cannot be avoided by simply removing it from the list of axioms (see Hrbacek [13]). These theories postulate some version of the *Standardization Principle*:

[1] Detailed examination of Connes's views is carried out in Kanovei, Katz and Mormann [22], Katz and Leichtnam [25] and Sanders [31].

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> For every formula $\Phi(x)$ in the language of the theory (possibly with additional parameters) and every standard set A there exists a standard set S such that for all standard x $x \in S \leftrightarrow x \in A \wedge \Phi(x)$.

This set is denoted ${}^{\mathbf{st}}\{x \in A \mid \Phi(x)\}$. It follows that, for an unlimited $\nu \in \mathbb{N}$, the standard set $U = {}^{\mathbf{st}}\{X \in \mathcal{P}(\mathbb{N}) \mid \nu \in X\}$ is a nonprincipal ultrafilter over $\mathbb{N}$.

All work in Nonstandard Analysis based on these two familiar frameworks thus depends essentially on the Axiom of Choice.[2]

While strong forms of **AC** (Zorn's Lemma, Prime Ideal Theorem) are instrumental in many abstract areas of mathematics, such as General Topology (the product of compact spaces is compact), Measure Theory (there exist sets that are not Lebesgue measurable) or Functional Analysis (Hahn–Banach theorem), it is undesirable to have to rely on them for results in "ordinary" mathematics such as Calculus, finite Combinatorics and Number Theory.[3]

Hrbacek and Katz [15] introduced nonstandard set theories **SPOT** and **SCOT**. In order to avoid the reliance on **AC**, Standardization needs to be weakened. The theory **SPOT** has three simple axioms: Standard Part, Nontriviality and Transfer. It is a subtheory of the better known nonstandard set theories **IST** and **HST**, but unlike them, it is a conservative extension of **ZF**. Arguments carried out in **SPOT** thus do not depend on any form of **AC**. Infinitesimal analysis can be conducted in **SPOT** in the usual way. It only needs to be verified that any use of Standardization can be justified by the special cases of this principle that are available in **SPOT**.

Traditional proofs in "ordinary" mathematics either do not use **AC** at all, or refer only to its weak forms, notably the Axiom of Countable Choice (**ACC**) or the stronger Axiom of Dependent Choice (**ADC**). These axioms are generally accepted and often used without comment. They are necessary to prove eg the equivalence of the ε-δ definition and the sequential definition of continuity at a given point for functions $f\colon X \subseteq \mathbb{R} \to \mathbb{R}$, or the countable additivity of Lebesgue measure, but they do not

[2] Nonstandard Analysis that does not use **AC**, or uses only weak versions of it, can be found in the work of Chuaqui, Sommer and Suppes (see eg [33]), in papers on Reverse Mathematics of Nonstandard Analysis (eg Keisler [27], Sanders [30], van den Berg et al [3] and others), and in the work of Hrbacek and Katz [15, 16, 17] and the present paper, based on **SPOT**/**SCOT**.

[3] The issue is not the validity of such results but the method of proof. It is a consequence of Shoenfield's Absoluteness Theorem (Jech [19, Theorem 98]) that all Π^1_4 sentences of second-order arithmetic that are provable in **ZFC** are also provable in **ZF**. Most theorems of Number Theory and Real Analysis (eg Peano's Theorem; see Hanson's answer in [10]) can be formalized as Π^1_4 statements. But the **ZF** proofs obtained from **ZFC** proofs by this method are far from "ordinary."

imply the strong consequences of **AC** such as the existence of nonprincipal ultrafilters or the Banach–Tarski paradox. The theory **SCOT** is obtained by adding to **SPOT** the external version of the Axiom of Dependent Choice; it is a conservative extension of **ZF** + **ADC**.

Nonstandard Analysis with *multiple levels of standardness* has been used in combinatorics and number theory by Renling Jin, Terence Tao, Mauro Di Nasso and others. Jin [20] recently gave a groundbreaking nonstandard proof of Szemerédi's Theorem in a model-theoretic framework that has three levels of infinity.

Szemerédi's Theorem *If $D \subseteq \mathbb{N}$ has a positive upper density, then D contains a k–term arithmetic progression for every $k \in \mathbb{N}$.*

The main objective of this paper is to extend the above two theories to theories **SPOTS** and **SCOTS** with many levels of standardness and consider their relationship to **AC**. An outline of **SPOT** and **SCOT** is given in Section 2. Section 3 reviews the familiar properties of ultrapowers and iterated ultrapowers in a form suitable to motivate multi-level versions of these theories, which are formulated in Section 4. The next three sections illustrate various ways to use multiple levels of standardness. In Section 5 Jin's proof of Ramsey's Theorem is formalized in **SPOTS**, and Section 6 explains how Jin's proof of Szemerédi's Theorem can be developed in it. The multi-level nonstandard approach to Calculus employed in Hrbacek, Lessmann and O'Donovan [14] can also be formalized in **SPOTS** and thus does not require any more **AC** than the traditional approach; this is shown in Section 7. Finally, in Section 8 it is proved that **SCOTS** is a conservative extension of **ZF** + **ADC** and that **SPOTS** is conservative over **ZF** + **ACC**. It is an open problem whether **SPOTS** is a conservative extension of **ZF**.

2 Theories SPOT and SCOT

By an $\in$–language we mean the language that has a primitive binary membership predicate $\in$. The classical theories **ZF** and **ZFC** are formulated in the $\in$–language. It is enriched by defined symbols for constants, relations, functions and operations customary in traditional mathematics. For example, it contains names $\mathbb{N}$ and $\mathbb{R}$ for the sets of natural and real numbers; they are viewed as defined in the traditional way ($\mathbb{N}$ is the least inductive set, $\mathbb{R}$ is defined in terms of Dedekind cuts or Cauchy sequences).

Nonstandard set theories, including **SPOT** and **SCOT**, are formulated in the **st**-$\in$–language. They add to the $\in$–language a unary predicate symbol **st**, where **st**(x) reads

"x is standard," and possibly other symbols. They postulate that standard infinite sets contain also nonstandard elements. For example, $\mathbb{R}$ contains infinitesimals and unlimited reals, and $\mathbb{N}$ contains unlimited natural numbers.

We use $\forall$ and $\exists$ as quantifiers over sets and $\forall^{\mathbf{st}}$ and $\exists^{\mathbf{st}}$ as quantifiers over standard sets. All free variables of a formula $\Phi(v_1, \ldots, v_k)$ are assumed to be among $v_1, \ldots, v_k$ *unless* explicitly specified otherwise. This is usually done informally by saying that the formula has *parameters* (ie, additional free variables), possibly restricted to objects of a certain kind).

The axioms of **SPOT** are:

ZF (Zermelo–Fraenkel Set Theory; Separation and Replacement schemata apply to $\in$–formulas only.)

T (Transfer) Let $\phi(\nu)$ be an $\in$–formula with standard parameters. Then:

$$\forall^{\mathbf{st}} x\ \phi(x) \to \forall x\ \phi(x)$$

O (Nontriviality) $\exists \nu \in \mathbb{N}\ \forall^{\mathbf{st}} n \in \mathbb{N}\ (n \neq \nu)$

SP (Standard Part) $\forall A \subseteq \mathbb{N}\ \exists^{\mathbf{st}} B \subseteq \mathbb{N}\ \forall^{\mathbf{st}} n \in \mathbb{N}\ (n \in B\ \leftrightarrow\ n \in A)$

We state some general results provable in **SPOT** (Hrbacek and Katz [15]).

Proposition 2.1 Standard natural numbers precede all nonstandard natural numbers:

$$\forall^{\mathbf{st}} n \in \mathbb{N}\ \forall m \in \mathbb{N}\ (m < n \to \mathbf{st}(m))$$

Proposition 2.2 (Countable Idealization) *Let $\phi(u, v)$ be an $\in$–formula with arbitrary parameters. Then:*

$$\forall^{\mathbf{st}} n \in \mathbb{N}\ \exists x\ \forall m \in \mathbb{N}\ (m \leq n\ \to \phi(m, x))\ \leftrightarrow\ \exists x\ \forall^{\mathbf{st}} n \in \mathbb{N}\ \phi(n, x)$$

The dual form of Countable Idealization is:

$$\exists^{\mathbf{st}} n \in \mathbb{N}\ \forall x\ \exists m \in \mathbb{N}\ (m \leq n\ \wedge\ \phi(m, x))\ \leftrightarrow\ \forall x\ \exists^{\mathbf{st}} n \in \mathbb{N}\ \phi(n, x)$$

Countable Idealization easily implies the following more familiar form. We use $\forall^{\mathbf{st\,fin}}$ and $\exists^{\mathbf{st\,fin}}$ as quantifiers over standard finite sets.

Let $\phi(u, v)$ be an $\in$–formula with arbitrary parameters. For every standard countable set A,

$$\forall^{\mathbf{st\,fin}} a \subseteq A\ \exists x\ \forall y \in a\ \phi(x, y)\ \leftrightarrow\ \exists x\ \forall^{\mathbf{st}} y \in A\ \phi(x, y).$$

The axiom **SP** is often used in the equivalent form:

(**SP′**) $$\forall x \in \mathbb{R}\ (x \text{ limited } \rightarrow \exists^{\mathbf{st}} r \in \mathbb{R}\ (x \simeq r))$$

We recall that x is *limited* if and only if $|x| \leq n$ for some standard $n \in \mathbb{N}$, and $x \simeq r$ if and only if $|x - r| \leq 1/n$ for all standard $n \in \mathbb{N}$, $n \neq 0$; x is *infinitesimal* if $x \simeq 0 \wedge x \neq 0$. The unique standard real number r is called the *standard part of* x or the *shadow of* x; notation $r = \mathbf{sh}(x)$.

The axiom **SP** is also equivalent to Standardization over countable sets for $\in$–formulas (with arbitrary parameters):

Let $\phi(v)$ be an $\in$–formula with arbitrary parameters. Then:

(**SP″**) $$\exists^{\mathbf{st}} S\ \forall^{\mathbf{st}} n\ (n \in S \ \leftrightarrow\ n \in \mathbb{N} \wedge \phi(n))$$

Proof Let $A = \{n \in \mathbb{N} \mid \phi(n)\}$ and apply **SP**. □

The "nonstandard" axioms of **SPOT** extend to **ZF** the insights of Leibniz about real numbers (see Bair et al [1, 2], Katz and Sherry [26] and Katz, Kuhlemann and Sherry [24]):

- Assignable vs inassignable distinction [standard vs nonstandard]
- Law of continuity [Transfer]
- Existence of infinitesimals [Nontriviality]
- Equality up to infinitesimal terms that need to be discarded [Standard Part]

This can be taken as a justification of the axioms of **SPOT** independent of the proof of its conservativity over **ZF**.

The scope of the axiom schema **SP″** can be extended.

Definition 2.3 An **st**-$\in$–formula $\Phi(v_1, \ldots, v_r)$ is *special* if it is of the form

$$\mathsf{Q}^{\mathbf{st}} u_1 \ldots \mathsf{Q}^{\mathbf{st}} u_s\ \psi(u_1, \ldots, u_s, v_1, \ldots, v_r)$$

where ψ is an $\in$–formula and each Q stands for $\exists$ or $\forall$.

We use $\forall^{\mathbf{st}}_{\mathbb{N}} u \ldots$ and $\exists^{\mathbf{st}}_{\mathbb{N}} u \ldots$ as shorthand for, respectively, $\forall^{\mathbf{st}} u\,(u \in \mathbb{N} \rightarrow \ldots)$ and $\exists^{\mathbf{st}} u\,(u \in \mathbb{N} \wedge \ldots)$.

An $\mathbb{N}$-*special* formula is a formula of the form

$$\mathsf{Q}^{\mathbf{st}}_{\mathbb{N}} u_1 \ldots \mathsf{Q}^{\mathbf{st}}_{\mathbb{N}} u_s\ \psi(u_1, \ldots u_s, v_1, \ldots, v_r)$$

where ψ is an $\in$–formula.

Proposition 2.4 (**SPOT**) (Countable Standardization for $\mathbb{N}$–Special Formulas) *Let $\Phi(v)$ be an $\mathbb{N}$–special formula with arbitrary parameters. Then:*

$$\exists^{\mathbf{st}} S\, \forall^{\mathbf{st}} n\, (n \in S \;\leftrightarrow\; n \in \mathbb{N} \wedge \Phi(n))$$

Of course, $\mathbb{N}$ can be replaced by any standard countable set.

Proof We give the argument for a typical case:

$$\forall^{\mathbf{st}}_{\mathbb{N}} u_1\, \exists^{\mathbf{st}}_{\mathbb{N}} u_2\, \forall^{\mathbf{st}}_{\mathbb{N}} u_3\; \psi(u_1, u_2, u_3, v)$$

By $\mathbf{SP}''$ there is a standard set R such that for all standard n_1, n_2, n_3, n :

$$\langle n_1, n_2, n_3, n\rangle \in R \;\leftrightarrow\; \langle n_1, n_2, n_3, n\rangle \in \mathbb{N}^4 \wedge \psi(n_1, n_2, n_3, n)$$

We let $R_{n_1,n_2,n_3} = \{n \in \mathbb{N} \mid \langle n_1, n_2, n_3, n\rangle \in R\}$ and let:

$$S = \bigcap_{n_1 \in \mathbb{N}} \bigcup_{n_2 \in \mathbb{N}} \bigcap_{n_3 \in \mathbb{N}} R_{n_1,n_2,n_3}$$

Then S is standard and for all standard n:

$$\begin{aligned} n \in S \;&\leftrightarrow\; \forall n_1 \in \mathbb{N}\, \exists n_2 \in \mathbb{N}\, \forall n_3 \in \mathbb{N}\; (n \in R_{n_1,n_2,n_3}) \\ &\leftrightarrow\; \text{(by Transfer) } \forall^{\mathbf{st}}_{\mathbb{N}} n_1\, \exists^{\mathbf{st}}_{\mathbb{N}} n_2\, \forall^{\mathbf{st}}_{\mathbb{N}} n_3\; (n \in R_{n_1,n_2,n_3}) \\ &\leftrightarrow\; \text{(by definition of } R) \; \forall^{\mathbf{st}}_{\mathbb{N}} n_1\, \exists^{\mathbf{st}}_{\mathbb{N}} n_2\, \forall^{\mathbf{st}}_{\mathbb{N}} n_3\; \psi(n_1, n_2, n_3, n) \\ &\leftrightarrow\; \Phi(n). \quad \square \end{aligned}$$

Infinitesimal calculus can be developed in **SPOT** as far as the global version of Peano's Theorem; see Hrbacek and Katz [16, 17].

Peano's Theorem *Let $F\colon [0,\infty) \times \mathbb{R} \to \mathbb{R}$ be a continuous function. There is an interval $[0, a)$ with $0 < a \leq \infty$ and a function $y\colon [0, a) \to \mathbb{R}$ such that*

$$y(0) = 0, \qquad y'(x) = F(x, y(x))$$

holds for all $x \in [0, a)$, and if $a \in \mathbb{R}$ then $\lim_{x \to a^-} y(x) = \pm\infty$.

We note that traditional proofs of the global version of Peano's Theorem use Zorn's Lemma or the Axiom of Dependent Choice.

It is useful to extend **SPOT** by two additional special cases of Standardization.

SN (Standardization for **st**-$\in$–formulas with no parameters or, equivalently, with only standard parameters) Let $\Phi(v)$ be an **st**-$\in$–formula with standard parameters. Then:

$$\forall^{\mathbf{st}} A\, \exists^{\mathbf{st}} S\, \forall^{\mathbf{st}} x\; (x \in S \;\leftrightarrow\; x \in A \wedge \Phi(x))$$

SF (Standardization over standard finite sets) Let $\Phi(v)$ be an **st**-$\in$–formula with arbitrary parameters. Then:

$$\forall^{\mathbf{st\,fin}} A\ \exists^{\mathbf{st}} S\ \forall^{\mathbf{st}} x\ (x \in S\ \leftrightarrow\ x \in A\ \wedge\ \Phi(x))$$

An important consequence of **SF** is the ability to carry out external induction.

Proposition 2.5 (External Induction) *Let $\Phi(v)$ be an* **st**-$\in$*–formula with arbitrary parameters. Then* **SPOT** + **SF** *proves the following:*

$$[\,\Phi(0)\ \wedge\ \forall^{\mathbf{st}} n \in \mathbb{N}\,(\Phi(n) \to \Phi(n+1))\,] \to \forall^{\mathbf{st}} m\,\Phi(m)$$

Proof Let $m \in \mathbb{N}$ be standard. If $m = 0$, then $\Phi(m)$ holds. Otherwise **SF** yields a standard set $S \subseteq m$ such that $\forall^{\mathbf{st}} n < m\,(n \in S\ \leftrightarrow\ \Phi(n))$; clearly $0 \in S$. As S is finite, it has a greatest element k, which is standard by Transfer. If $k < m$, then $k+1 \in S$, a contradiction. Hence $k = m$ and $\Phi(m)$ holds. □

SPOT$^+$ is **SPOT + SN + SF**.

The theory **SPOT**$^+$ *is a conservative extension of* **ZF**.

This is proved for **SPOT** + **SN** in Hrbacek and Katz [15, Theorem B] (Propositions 4.15 and 6.7 there). The argument that **SF** can be also added conservatively over **ZF** is given at the end of Section 8 (Proposition 8.7).

The theory **SCOT** is **SPOT**$^+$ + **DC**, where:

DC (Dependent Choice for **st**-$\in$–formulas) Let $\Phi(u,v)$ be an **st**-$\in$–formula with arbitrary parameters. If $\forall x\,\exists y\ \Phi(x,y)$, then for any b there is a sequence $\langle b_n \mid n \in \mathbb{N}\rangle$ such that $b_0 = b$ and $\forall^{\mathbf{st}} n \in \mathbb{N}\ \Phi(b_n, b_{n+1})$.

Some general consequences of **SCOT** are (see [15]):

CC (Countable **st**-$\in$–Choice) Let $\Phi(u,v)$ be an **st**-$\in$–formula with arbitrary parameters. Then:

$$\forall^{\mathbf{st}} n \in \mathbb{N}\ \exists x\ \Phi(n,x) \to \exists f\ (f \text{ is a function}\ \wedge\ \forall^{\mathbf{st}} n \in \mathbb{N}\ \Phi(n, f(n))$$

SC (Countable Standardization) Let $\Psi(v)$ be an **st**-$\in$–formula with arbitrary parameters. Then:

$$\exists^{\mathbf{st}} S\ \forall^{\mathbf{st}} x\ (x \in S\ \leftrightarrow\ x \in \mathbb{N}\ \wedge\ \Psi(x))$$

SCOT is a conservative extension of **ZF** + **ADC** [15, Theorem 5.10].[4] It allows such features as an infinitesimal construction of the Lebesgue measure. It implies the axioms of Nelson's *Radically Elementary Probability Theory* [28].

[4] It is an open question whether **SPOT**$^+$ + **SC** is a conservative extension of **ZF**.

3 Ultrafilters, ultrapowers and iterated ultrapowers

In this section we review the construction of iterated ultrapowers in a form suitable for motivation and establishing the conservativity of the theories formulated in Section 4. We assume **ZFC**, use classes freely and give no proofs. Some references for this material are Chang and Keisler [4], Enayat et al [9] and Hrbacek [11, 12].

Model theory deals with structures that are sets. For our purposes we need to construct ultrapowers of the entire set-theoretic universe $\mathbb{V}$. That means we have to deal with structures that are proper classes, eg $(\mathbb{V}, \in)$. We sometimes use the model-theoretic language and say that such structure *satisfies a formula*, or that a mapping is an *elementary embedding* of one class structure into another. It is well-known that the satisfaction relation $\vDash$ for such structures cannot be proved to exist in **ZFC**. But the concept makes sense for any one particular formula. Thus, if $\mathfrak{U} = (\mathbb{U}; \mathbb{V}_1, \mathbb{V}_2, \ldots)$ is a (class) structure and $\Phi(v_1, \ldots, v_r)$ is a formula in the language of $\mathfrak{U}$, we write $\Phi^{\mathbb{U}}(v_1, \ldots, v_r)$ for the formula obtained from Φ by restricting all quantifiers to $\mathbb{U}$, ie, by replacing each occurrence of $\forall v$ by $\forall v \in \mathbb{U}$ and each occurrence of $\exists v$ by $\exists v \in \mathbb{U}$. (We usually abuse notation by not distinguishing between classes and their names in the language of $\mathfrak{U}$.) The statement that $\mathbf{J}$ is an elementary embedding of $\mathfrak{U}^1$ to $\mathfrak{U}^2$, for example, means that, given any formula Φ of the appropriate language,

$$\forall x_1, \ldots, x_r \in \mathbb{U}^1 \, (\Phi^{\mathbb{U}^1}(x_1, \ldots, x_r) \;\leftrightarrow\; \Phi^{\mathbb{U}^2}(\mathbf{J}(x_1), \ldots, \mathbf{J}(x_r))).$$

Let U be an ultrafilter over I. For $f, g \in \mathbb{V}^I$ we define:

$$f =_U g \text{ if and only if } \{i \in I \mid f(i) = g(i)\} \in U$$
$$f \in_U g \text{ if and only if } \{i \in I \mid f(i) \in g(i)\} \in U$$

The usual procedure at this point is to form equivalence classes $[f]_U$ of functions $f \in \mathbb{V}^I$ modulo $=_U$, using "Scott's trick" of taking only the functions of the minimal von Neumann rank to guarantee that the equivalence classes are sets:

$$[f]_U = \{g \in \mathbb{V}^I \mid g =_U f \text{ and } \forall h \in \mathbb{V}^I \, (h =_U f \to \operatorname{rank} h \geq \operatorname{rank} g)\}$$

see Jech [19, (9.3) and (28.15)]. Then $\mathbb{V}^I/U = \{[f]_U \mid f \in \mathbb{V}^I\}$, and $[f]_U \in_U [g]_U$ if and only if $f \in_U g$.

The *ultrapower* of $\mathbb{V}$ by U is the structure $(\mathbb{V}^I/U, \in_U)$.

Let $\pi : I \to J$. Define the ultrafilter $V = \pi[U]$ over J by:

$$\pi[U] = \{Y \subseteq J \mid \pi^{-1}[Y] \in U\}$$

The mapping π induces $\Pi : \mathbb{V}^J/V \to \mathbb{V}^I/U$ by $\Pi([g]_V) = [g \circ \pi]_U$.

The following proposition is an easy consequence of Łoś's Theorem.

Proposition 3.1 *The mapping* Π *is well-defined and it is an elementary embedding of* $(\mathbb{V}^J/V, \in_V)$ *into* $(\mathbb{V}^I/U, \in_U)$.

In detail: for any $\in$*–formula* ϕ *and all* $[f_1]_V, \ldots [f_r]_V \in \mathbb{V}^J/V$,

$$\phi^{\mathbb{V}^J/V}([f_1]_V, \ldots, [f_r]_V) \;\leftrightarrow\; \phi^{\mathbb{V}^I/U}(\Pi([f_1]_V), \ldots, \Pi([f_r])_V).$$

The *tensor product* of ultrafilters U and V, respectively over I and J, is the ultrafilter over $I \times J$ defined by (note the order; Chang and Keisler [4] use the opposite order):

$$Z \in U \otimes V \text{ if and only if } \{x \in I \mid \{y \in J \mid \langle x, y\rangle \in Z\} \in V\} \in U$$

The *n-th tensor power* of U is the ultrafilter over I^n defined by recursion:

$$\otimes^0 U = \{\{\emptyset\}\}; \quad \otimes^1 U = U; \quad \otimes^{n+1} U = U \otimes (\otimes^n U)$$

In the following, a, b range over finite subsets of $\mathbb{N}$.

If $|\mathsf{a}| = n$, let π be the mapping of I^n onto I^{a} induced by the order-preserving mapping of n onto a. It follows that $U_{\mathsf{a}} = \pi[\otimes^n U]$ is an ultrafilter over I^{a}.

For $\mathsf{a} \subseteq \mathsf{b}$ let $\pi^{\mathsf{b}}_{\mathsf{a}}$ be the restriction map of I^{b} onto I^{a} defined by $\pi^{\mathsf{b}}_{\mathsf{a}}(\mathbf{i}) = \mathbf{i} \restriction \mathsf{a}$ for $\mathbf{i} \in I^{\mathsf{b}}$.

It is easy to see that $U_{\mathsf{a}} = \pi^{\mathsf{b}}_{\mathsf{a}}[U_{\mathsf{b}}]$. We let $\mathbb{V}_{\mathsf{a}} = \mathbb{V}^{I^{\mathsf{a}}}/U_{\mathsf{a}}$ and write $[f]_{\mathsf{a}}$ for $[f]_{U_{\mathsf{a}}}$ and $\in_{\mathsf{a}}$ for $\in_{U_{\mathsf{a}}}$. The mapping $\Pi^{\mathsf{b}}_{\mathsf{a}}$ induced by $\pi^{\mathsf{b}}_{\mathsf{a}}$ is an elementary embedding of $(\mathbb{V}_{\mathsf{a}}, \in_{\mathsf{a}})$ into $(\mathbb{V}_{\mathsf{b}}, \in_{\mathsf{b}})$.

Proposition 3.2 *If* $f \in \mathbb{V}^{I^{\mathsf{a}}}$, $g \in \mathbb{V}^{I^{\mathsf{b}}}$ *and* $\Pi^{\mathsf{a}\cup\mathsf{b}}_{\mathsf{a}}([f]_{\mathsf{a}}) = \Pi^{\mathsf{a}\cup\mathsf{b}}_{\mathsf{b}}([g]_{\mathsf{b}})$, *then there is* $h \in \mathbb{V}^{I^{\mathsf{a}\cap\mathsf{b}}}$ *such that* $\Pi^{\mathsf{a}\cup\mathsf{b}}_{\mathsf{a}\cap\mathsf{b}}([h]_{\mathsf{a}\cap\mathsf{b}}) = \Pi^{\mathsf{a}\cup\mathsf{b}}_{\mathsf{a}}([f]_{\mathsf{a}}) = \Pi^{\mathsf{a}\cup\mathsf{b}}_{\mathsf{b}}([g]_{\mathsf{b}})$.

Let $f, g \in \bigcup_{\mathsf{a}} \mathbb{V}^{I^{\mathsf{a}}}$; say $f \in \mathbb{V}^{I^{\mathsf{a}}}$ and $g \in \mathbb{V}^{I^{\mathsf{b}}}$. We define:

$$f =_\infty g \text{ if and only if } f \circ \pi^{\mathsf{a}\cup\mathsf{b}}_{\mathsf{a}} =_{U_{\mathsf{a}\cup\mathsf{b}}} g \circ \pi^{\mathsf{a}\cup\mathsf{b}}_{\mathsf{b}}$$
$$f \in_\infty g \text{ if and only if } f \circ \pi^{\mathsf{a}\cup\mathsf{b}}_{\mathsf{a}} \in_{U_{\mathsf{a}\cup\mathsf{b}}} g \circ \pi^{\mathsf{a}\cup\mathsf{b}}_{\mathsf{b}}$$

We let $[f]_\infty$ be the equivalence class of f modulo $=_\infty$ (again using Scott's Trick), and let $\mathbb{V}_\infty = \{[f]_\infty \mid f \in \bigcup_{\mathsf{a}} \mathbb{V}^{I^{\mathsf{a}}}\}$ and $[f]_\infty \in_\infty [g]_\infty$ if and only if $f \in_\infty g$.

The *iterated ultrapower of* $\mathbb{V}$ *by* U is the structure $(\mathbb{V}_\infty, \in_\infty)$. It is the direct limit of the system of structures $(\mathbb{V}_{\mathsf{a}},\ \Pi^{\mathsf{b}}_{\mathsf{a}};\ \mathsf{a}, \mathsf{b} \in \mathcal{P}^{\mathbf{fin}}(\mathbb{N}),\ \mathsf{a} \subseteq \mathsf{b})$. The mappings $\Pi^\infty_{\mathsf{a}} :\ \mathbb{V}_{\mathsf{a}} \to \mathbb{V}_\infty$

defined by $\Pi^{\infty}_{\mathsf{a}}([f]_{\mathsf{a}}) = [f]_{\infty}$ are elementary embeddings; we identify $\mathbb{V}_{\mathsf{a}}$ with its image by this embedding. In particular, $\Pi^{\infty}_{\emptyset}$ embeds $\mathbb{V} \cong \mathbb{V}_{\emptyset}$ elementarily into $\mathbb{V}_{\infty}$.

In addition to the canonical elementary embeddings $\Pi^{\mathsf{b}}_{\mathsf{a}}$ for $\mathsf{a} \subseteq \mathsf{b}$, the iterated ultrapower allows other elementary embeddings, due to the fact that the same ultrafilter U is used throughout the construction. If $|\mathsf{a}| = |\mathsf{b}|$ and α is the order-preserving mapping of a onto b, define $\pi^{\mathsf{b}}_{\mathsf{a}} : I^{\mathsf{b}} \to I^{\mathsf{a}}$ by $\pi^{\mathsf{b}}_{\mathsf{a}}(\mathbf{i}) = \mathbf{i} \circ \alpha$ for $\mathbf{i} \in I^{\mathsf{b}}$. Then $\Pi^{\mathsf{b}}_{\mathsf{a}}$ is an isomorphism of $(\mathbb{V}_{\mathsf{a}}, \in_{\mathsf{a}})$ and $(\mathbb{V}_{\mathsf{b}}, \in_{\mathsf{b}})$.

We fix $r \in \mathbb{N}$. For $f \in \mathbb{V}^{I^{r+n}}$ define $f|I^r : I^r \to \mathbb{V}^{I^n}$ by $f|I^r(\mathbf{i}) = f_{\mathbf{i}}$ where $f_{\mathbf{i}}(\mathbf{j}) = f(\mathbf{i}, \mathbf{j})$ for all $\mathbf{j} \in I^n$. For $[f]_{\infty} \in \mathbb{V}_{\infty}$, say $f \in \mathbb{V}^{I^{r+n}}$, we let $\Omega([f]_{\infty}) = [F]_{U_r}$ where $F(\mathbf{i}) = [f_{\mathbf{i}}]_{\infty}$ for all $\mathbf{i} \in I^r$. It is routine to check that $\Omega : \mathbb{V}_{\infty} \to (\mathbb{V}_{\infty})^{I^r}/U_r$ is well-defined.

We use the notations $r \oplus \mathsf{a} = \{r + s \mid s \in \mathsf{a}\}$ and $r \boxplus \mathsf{a} = r \cup (r \oplus \mathsf{a})$. Note that if $\mathsf{a} = n = \{0, \ldots, n-1\} \in \mathbb{N}$, then $r \boxplus n = r + n$.

Proposition 3.3 *(Factoring Lemma) The mapping Ω is an isomorphism of the structures*

$$(\mathbb{V}_{\infty}, \in_{\infty}, \mathbb{V}_{r\boxplus\mathsf{a}}, \Pi^{r\boxplus\mathsf{b}}_{r\boxplus\mathsf{a}};\ \mathsf{a}, \mathsf{b} \in \mathcal{P}^{\mathbf{fin}}(\mathbb{N}), |\mathsf{a}| = |\mathsf{b}|)$$

and

$$(\mathbb{V}_{\infty}, \in_{\infty}, \mathbb{V}_{\mathsf{a}}, \Pi^{\mathsf{b}}_{\mathsf{a}};\ \mathsf{a}, \mathsf{b} \in \mathcal{P}^{\mathbf{fin}}(\mathbb{N}), |\mathsf{a}| = |\mathsf{b}|)^{I^r}/U_r.$$

4 SPOTS

Theories with many levels of standardness have been developed in Péraire, [29] (**RIST**) and Hrbacek [11, 12] (**GRIST**). The characteristic feature of these theories is that the unary standardness predicate $\mathbf{st}(v)$ is subsumed under a binary *relative standardness* predicate $\mathbf{sr}(u, v)$.

The main advantage of theories with many levels of standardness is that nonstandard methods can be applied to arbitrary objects, not just the standard ones. For example, the nonstandard definition of the derivative

$$f'(a) = \mathbf{sh}\left(\frac{f(a+h) - f(a)}{h}\right) \quad \text{where } h \text{ is infinitesimal}$$

which in a single-level nonstandard analysis works for standard f and a only, in these theories works for all f and a, provided "infinitesimal" is understood as "infinitesimal relative to the level of f and a" and "**sh**" is "**sh** relative to the level of f and a." In the book Hrbacek, Lessmann and O'Donovan [14] this approach is used to develop elementary calculus.

Jin's work using multi-level nonstandard analysis goes beyond the features postulated by these theories in that it also employs nontrivial elementary embeddings (ie, other than those provided by inclusion of one level in a higher level).

The language of **SPOTS** has a binary predicate symbol $\in$, a binary predicate symbol **sr** (**sr**(u,v) reads "v is *u-standard*") and a ternary function symbol **ir** that captures the relevant isomorphisms. The unary predicate **st**(v) stands for **sr**$(\emptyset,v)$, Variables a, b (with decorations) range over *standard finite* subsets of $\mathbb{N}$; we refer to them as *labels*. We use the class notation $\mathbb{S}_\mathsf{a} = \{x \mid \mathbf{sr}(\mathsf{a},x)\}$ and $\mathbf{I}^\mathsf{b}_\mathsf{a} = \{\langle x,y\rangle \mid \mathbf{ir}(\mathsf{a},\mathsf{b},x) = y\}$. If a is a standard natural number, we use n instead of a; analogously for b and m. We call $\mathbb{S}_\mathsf{n}$ the n*–th level of standardness*. In particular, $\mathbb{S} = \mathbb{S}_0 = \{x \mid \mathbf{st}(x)\}$ is the universe of standard sets.

As in Section 3, for standard $r \in \mathbb{N}$ we let $r\oplus\mathsf{a} = \{r+s \mid s \in \mathsf{a}\}$ and $r\boxplus\mathsf{a} = r \cup (r\oplus\mathsf{a})$. Also $\mathsf{a} < \mathsf{b}$ stands for $\forall s \in \mathsf{a}\ \forall t \in \mathsf{b}\ (s < t)$.

A formula Φ is *admissible* if labels appear in it only as subscripts and superscripts of $\mathbb{S}$ and $\mathbf{I}$.

Definition 4.1 (*Admissible formulas*)

- $u = v$, $u \in v$, $v \in \mathbb{S}_\mathsf{a}$ and $\mathbf{I}^\mathsf{b}_\mathsf{a}(u) = v$ are admissible formulas
- If Φ and Ψ are admissible, then $\neg\Phi$, $\Phi \wedge \Psi$, $\Phi \vee \Psi$, $\Phi \to \Psi$ and $\Phi \leftrightarrow \Psi$ are admissible
- If Φ is admissible, then $\forall v\,\Phi$ and $\exists v\,\Phi$ are admissible

Let $\Phi^{\uparrow r}$ be the formula obtained from the admissible formula Φ by replacing each occurrence of every $\mathbb{S}_\mathsf{a}$ with $\mathbb{S}_{r\boxplus\mathsf{a}}$ and each occurrence of $\mathbf{I}^\mathsf{b}_\mathsf{a}$ with $\mathbf{I}^{r\boxplus\mathsf{b}}_{r\boxplus\mathsf{a}}$. In particular, if Φ is a formula where only the symbols $\mathbb{S}_\mathsf{n}$ and $\mathbf{I}^\mathsf{m}_\mathsf{n}$ for $\mathsf{n},\mathsf{m} \in \mathbb{N}\cap\mathbb{S}_0$ occur, then $\Phi^{\uparrow r}$ is obtained from Φ by shifting all levels by r. This is the special case that is most often used in practice.

The iterated ultrapower construction described in Section 3 suggests the axioms **IS**, **GT** and **HO**.

IS (Structural axioms)

(1) $\mathbf{sr}(u,v) \to \exists\mathsf{a}\ (u = \mathsf{a})$, $\mathbf{ir}(u,v,x) = y \to \exists\mathsf{a},\mathsf{b}\ (u = \mathsf{a} \wedge v = \mathsf{b})$, $\mathbf{I}^\mathsf{b}_\mathsf{a}(u) = v \to |\mathsf{a}| = |\mathsf{b}|$

(2) $\forall x\,\exists\mathsf{a}\ (x \in \mathbb{S}_\mathsf{a})$

(3) For all a,b, $\mathbb{S}_{\mathsf{a}\cap\mathsf{b}} = \mathbb{S}_\mathsf{a} \cap \mathbb{S}_\mathsf{b}$ (in particular, $\mathsf{a} \subseteq \mathsf{b} \to \mathbb{S}_\mathsf{a} \subseteq \mathbb{S}_\mathsf{b}$)

(4) If $|\mathsf{a}| = |\mathsf{a}'| = |\mathsf{a}''|$, then

$$\mathbf{I}^{\mathsf{a}'}_{\mathsf{a}} : \mathbb{S}_{\mathsf{a}} \to \mathbb{S}_{\mathsf{a}'}, \quad \mathbf{I}^{\mathsf{a}}_{\mathsf{a}} = \mathrm{Id}_{\mathbb{S}_{\mathsf{a}}}, \quad \mathbf{I}^{\mathsf{a}}_{\mathsf{a}'} = (\mathbf{I}^{\mathsf{a}'}_{\mathsf{a}})^{-1}, \quad \mathbf{I}^{\mathsf{a}'}_{\mathsf{a}} \circ \mathbf{I}^{\mathsf{a}''}_{\mathsf{a}'} = \mathbf{I}^{\mathsf{a}''}_{\mathsf{a}}$$

and $$\forall x, z \in \mathbb{S}_{\mathsf{a}} \; (x \in z \;\leftrightarrow\; \mathbf{I}^{\mathsf{a}'}_{\mathsf{a}}(x) \in \mathbf{I}^{\mathsf{a}'}_{\mathsf{a}}(z))$$

(5) If $|\mathsf{a}| = |\mathsf{a}'|$ and $\mathsf{b} \subset \mathsf{a}$, then

$$x \in \mathbb{S}_{\mathsf{b}} \to \mathbf{I}^{\mathsf{a}'}_{\mathsf{a}}(x) = \mathbf{I}^{\mathsf{b}'}_{\mathsf{b}}(x)$$

where b' is the image of b by the order-preserving map of a onto a'

GT (Generalized Transfer)

Let $\phi(v, v_1, \ldots, v_k)$ be an $\in$–formula. Then for all $\mathsf{a} \in \mathcal{P}^{\mathbf{fin}}(\mathbb{N}) \cap \mathbb{S}_0$

$$\forall x_1, \ldots, x_k \in \mathbb{S}_{\mathsf{a}} \; (\forall x \in \mathbb{S}_{\mathsf{a}} \; \phi(x, x_1, \ldots, x_k) \to \forall x \; \phi(x, x_1, \ldots, x_k)) \,.$$

HO (Homogeneous Shift)

Let $\Phi(v_1, \ldots, v_k)$ be an admissible formula. For all standard r and all $\mathsf{a} \in \mathcal{P}^{\mathbf{fin}}(\mathbb{N}) \cap \mathbb{S}_0$

$$\forall x_1, \ldots, x_k \in \mathbb{S}_{\mathsf{a}} \; [\, \Phi(x_1, \ldots, x_k) \;\leftrightarrow\; \Phi^{\uparrow r}(\mathbf{I}^{r \oplus \mathsf{a}}_{\mathsf{a}}(x_1), \ldots, \mathbf{I}^{r \oplus \mathsf{a}}_{\mathsf{a}}(x_k)) \,].$$

The language of **SPOTS** has an obvious interpretation in the iterated ultrapower described in Section 3: $\mathbb{S}_{\mathsf{a}}$ is interpreted as $\mathbb{V}^{I_{\mathsf{a}}}/U_{\mathsf{a}}$ and $\mathbf{I}^{\mathsf{b}}_{\mathsf{a}}$ is interpreted as $\Pi^{\mathsf{b}}_{\mathsf{a}}$.

Proposition 4.2 *Under the above interpretation, the axioms* **IS**, **GT** *and* **HO** *hold in the iterated ultrapower constructed in Section 3.*

Proof The axiom (3) in **IS** follows from Proposition 3.2; the rest is obvious.

Proposition 3.1 implies that, given $\mathsf{a} \subseteq \mathsf{b}$,

$$\forall x_1, \ldots, x_k \in \mathbb{S}_{\mathsf{a}} \; \left(\phi^{\mathbb{S}_{\mathsf{a}}}(x_1, \ldots, x_k) \;\leftrightarrow\; \phi^{\mathbb{S}_{\mathsf{b}}}(x_1, \ldots, x_k) \right).$$

By the Elementary Chain Theorem (Chang and Keisler [4], Theorem 3.1.13),

$$\forall x_1, \ldots, x_k \in \mathbb{S}_{\mathsf{a}} \; \left(\phi^{\mathbb{S}_{\mathsf{a}}}(x_1, \ldots, x_k) \;\leftrightarrow\; \phi(x_1, \ldots, x_k) \right).$$

The axiom **GT** is a special case.

HO is justified by the Factoring Lemma and Łoś's Theorem (specifically, by the fact that the canonical embedding of $(\mathbb{V}_\infty, \in_\infty, \mathbb{V}_{\mathsf{a}}, \Pi^{\mathsf{b}}_{\mathsf{a}})$ into its ultrapower by U_r is elementary). □

SPOTS is the theory $\mathbf{SPOT}^+ + \mathbf{IS} + \mathbf{GT} + \mathbf{HO}$, where Nontriviality is modified to $\exists \nu \in \mathbb{N} \cap \mathbb{S}_1\ (\forall^{\mathbf{st}} n \in \mathbb{N}\ (n \neq \nu)$ and **SN** and **SF** admit all formulas in the language of **SPOTS**.

A consequence of **GT** is the following proposition.

Proposition 4.3 *The mapping* $\mathbf{I}^{\mathsf{b}}_{\mathsf{a}}$ *is an elementary embedding of* $\mathbb{S}_{\mathsf{a}}$ *into* $\mathbb{I}$ *(where* $\mathbb{I}$ *is the class of all sets) and into* $\mathbb{S}_{\mathsf{b}'}$ *for every* $\mathsf{b}' \supseteq \mathsf{b}$.

An important consequence of **SPOTS** asserts that every natural number $k \in \mathbb{S}_{\mathsf{a}}$ is either standard or greater than all natural numbers at levels less than $\min \mathsf{a}$.

Proposition 4.4 (End Extension) *Let* $\mathsf{a} \neq \emptyset$ *and* $n = \min \mathsf{a} \in \mathbb{N}$. *Then:*

$$\forall k \in \mathbb{S}_{\mathsf{a}} \cap \mathbb{N}\ (k \in \mathbb{S}_0\ \vee\ \forall m \in \mathbb{S}_n\ (m < k))$$

Proof By Proposition 2.1, $\forall m \in \mathbb{N} \cap \mathbb{S}_0\ \forall k \in \mathbb{N}\ (k \leq m \to k \in \mathbb{S}_0)$. By **HO** this implies $\forall m \in \mathbb{N} \cap \mathbb{S}_n\ \forall k \in \mathbb{N}\ (k \leq m \to k \in \mathbb{S}_n)$. If $k \in \mathbb{S}_{\mathsf{a}} \cap \mathbb{N}$ and $\exists m \in \mathbb{S}_{\mathsf{n}}\ (m \geq k)$ then $k \in \mathbb{S}_n$ by the above. As $\mathsf{a} \cap n = \emptyset$, we get $k \in \mathbb{S}_0$. □

Let $\Phi(x_1, \ldots, x_k; \mathbb{S})$ denote a formula obtained from some **st**-$\in$–formula by replacing all occurences of $\mathbf{st}(\nu)$ with $\nu \in \mathbb{S}$, and let n, m be variables that do not occur in Φ and range over standard natural numbers.

Proposition 4.5 $\mathsf{n} \leq \mathsf{m} \to \forall x_1, \ldots, x_k \in \mathbb{S}_{\mathsf{n}}\ (\Phi(x_1, \ldots, x_k; \mathbb{S}_{\mathsf{n}})\ \leftrightarrow\ \Phi(x_1, \ldots, x_k; \mathbb{S}_{\mathsf{m}}))$

Proof Let $r = \mathsf{m} - \mathsf{n}$. By **HO** we have

$$\forall x_1, \ldots, x_k \in \mathbb{S}_0\ (\Phi(x_1, \ldots, x_k; \mathbb{S}_0)\ \leftrightarrow\ \Phi(x_1, \ldots, x_k; \mathbb{S}_r)).$$

(Note that $r \oplus 0 = 0$, $r \boxplus 0 = r$ and $\mathbf{I}^0_0 = \mathrm{Id}_{\mathbb{S}_0}$.) Then apply **HO** shift by n to this closed formula to obtain

$$\forall x_1, \ldots, x_k \in \mathbb{S}_{\mathsf{n}}\ (\Phi(x_1, \ldots, x_k; \mathbb{S}_n)\ \leftrightarrow\ \Phi(x_1, \ldots, x_k; \mathbb{S}_{\mathsf{m}})).$$ □

In particular, $\Phi(\mathbb{S}_0)$ implies $\Phi(\mathbb{S}_{\mathsf{n}})$ for every n. Hence the axioms of **SPOT**, postulated in **SPOTS** only about the level $\mathbb{S}_0$, hold there about every level $\mathbb{S}_{\mathsf{n}}$.

SCOTS is the theory $\mathbf{SCOT} + \mathbf{IS} + \mathbf{GT} + \mathbf{HO} = \mathbf{SPOTS} + \mathbf{DC}$, where the axiom schema **DC** is formulated as follows.

DC (Dependent Choice) Let $\Phi(u,v)$ be a formula in the language of **SPOTS**, with arbitrary parameters. For any a:

> If $\forall x \in \mathbb{S}_{\mathsf{a}}\, \exists y \in \mathbb{S}_{\mathsf{a}}\ \Phi(x,y)$, then for every $b \in \mathbb{S}_{\mathsf{a}}$ there is a sequence $\bar{b} = \langle b_n \mid n \in \mathbb{N}\rangle \in \mathbb{S}_{\mathsf{a}}$ such that $b_0 = b$ and $\forall n \in \mathbb{N} \cap \mathbb{S}_0\ \Phi(b_n, b_{n+1})$.

DC implies Countable Standardization (and hence **SF**).

SC (Countable Standardization) Let $\Psi(v)$ be a formula in the language of **SPOTS**, with arbitrary parameters. Then:

$$\exists S \in \mathbb{S}_0\ \forall x \in \mathbb{S}_0\ (x \in S\ \leftrightarrow\ x \in \mathbb{N}\ \wedge\ \Psi(x))$$

Theorem 4.6 **SCOTS** *is a conservative extension of* **ZF** + **ADC**.

Theorem 4.7 **SPOTS** *is conservative over* **ZF** + **ACC**.

The proofs are given in Section 8.

Conjecture **SPOTS** is a conservative extension of **ZF**.

5 Jin's proof of Ramsey's Theorem in SPOTS

Ramsey's Theorem *Given a coloring $c : [\mathbb{N}]^n \to r$ where $n, r \in \mathbb{N}$, there exists an infinite set $H \subseteq \mathbb{N}$ such that $c \restriction [H]^n$ is a constant function.*

We formalize in **SPOTS** the proof presented by Renling Jin [21] in his invited talk at the conference *Logical methods in Ramsey Theory and related topics*, Pisa, July 9 – 11, 2023. It is included here with his kind permission.

Proof It suffices to prove the theorem under the assumption that n, r, c are standard; the general result then follows by Transfer.

Let $\mathbf{I} = \mathbf{I}^{\{1,2,\ldots,n\}}_{\{0,1,\ldots,n-1\}}$. Fix $\nu \in \mathbb{N} \cap (\mathbb{S}_1 \setminus \mathbb{S}_0)$ and. define the n–tuple $\bar{x} = \langle x_1, \ldots, x_n\rangle$ by $x_1 = \nu,\ \ x_{i+1} = \mathbf{I}(x_i)$ for $i = 1, 2, \ldots, n-1$ (the existence of $\bar{x}$ is justified by **SF**). Let $c_0 = c(\bar{x})$.

Define a strictly increasing sequence $\{a_m\}^{\bullet}_{m=1} \subseteq \mathbb{N}$, where $\bullet \in \mathbb{N}$ or $\bullet = \infty$, recursively, using the notation $A_m = \{a_1, \ldots, a_m\}$ (also $a_0 = 0$ and $A_0 = \emptyset$):

$a_{m+1} =$ the least $a \in \mathbb{N}$ such that $a > a_m \wedge c \restriction [A_m \cup \{a\} \cup \bar{x}]^n = c_0$ if such a exists; otherwise a_{m+1} is undefined and the recursion stops.

Let $A = \bigcup_{m=1}^{\bullet} A_m$. Then A is a set and by **SP** there is a standard set H such that $\forall^{\mathbf{st}} x\, (x \in H \;\leftrightarrow\; x \in A)$. Clearly $c \restriction [H]^n = c_0$.

It remains to prove that H is infinite, ie, that a_m is defined and standard for all standard $m \in \mathbb{N} \setminus \{0\}$.

Fix a standard $m \in \mathbb{N}$. The sentence

$$\exists x \in \mathbb{N} \cap \mathbb{S}_1 \;\left(x > a_m \;\wedge\; c \restriction [A_m \cup \{x, \mathbf{I}(x_1), \ldots, \mathbf{I}(x_{n-1})\}]^n = c_0\right)$$

is true (just let $x = x_1$).

By **HO**, $\exists x \in \mathbb{N} \cap \mathbb{S}_0 \;\left(x > a_m \;\wedge\; c \restriction [A_m \cup \{x, x_1, \ldots, x_{n-1}\}]^n = c_0\right)$. Let a_{m+1} be the least such x and note that it is standard.

We have $c \restriction [A_{m+1} \cup \{x_1, \ldots, x_{n-1}\}]^n = c_0$. It remains to show that $c \restriction [A_{m+1} \cup \{x_1, \ldots, x_{n-1}, x_n\}]^n = c_0$.

Consider $\bar{b} = \{b_1 < \ldots < b_n\} \in [A_{m+1} \cup \{x_1, \ldots, x_{n-1}, x_n\}]^n$.

If $b_n < x_n$ then $b_n \leq x_{n-1}$ and $c(\bar{b}) = c_0$.

If $b_1 = x_1$ then $\bar{b} = \bar{x}$ and $c(\bar{b}) = c(\bar{x})) = c_0$.

Otherwise $b_1 \in \mathbb{N} \cap \mathbb{S}_0$ and $b_n = x_n$. Let p be the largest value such that $x_p \notin \bar{b}$ (clearly $1 \leq p < n$) and let $\mathbf{J} = \mathbf{I}^{\{0,\ldots,p-1,p+1,\ldots,x_n\}}_{\{0,\ldots,n-1\}}$.

Note that $\mathbf{J}(b_j) = b_j$ for $j \leq p$, $b_j = x_j$, and $\mathbf{J}(b_j) = \mathbf{J}(x_j) = x_{j+1}$ for $p < j \leq n-1$ (because $\mathbf{I}^{\{p+1,\ldots,n\}}_{\{p,\ldots,n-1\}} \subseteq \mathbf{I}, \mathbf{J}$, ie, $\mathbf{I}$ and $\mathbf{J}$ agree on $\mathbb{S}_{\{p,\ldots,n-1\}}$). Let $\bar{b}' = \mathbf{J}^{-1}(\bar{b})$. Then $\bar{b}' \in [A_{m+1} \cup \{x_1, \ldots, x_{n-1}\}]^n$, hence $c(\bar{b}') = c_0$. By **HO** shift via $\mathbf{J}$, $c(\bar{b}) = c_0$. □

6 Jin's proof of Szemerédi's Theorem in SPOTS

Jin's proof in [20] uses four universes ($\mathbb{V}_0, \mathbb{V}_1, \mathbb{V}_2$ and $\mathbb{V}_3$) and some additional elementary embeddings. Let $\mathbb{N}_j = \mathbb{N} \cap \mathbb{V}_j$ and $\mathbb{R}_j = \mathbb{R} \cap \mathbb{V}_j$ for $j = 0, 1, 2, 3$. Jin summarizes the required properties of these universes:

0. $\mathbb{V}_0 \prec \mathbb{V}_1 \prec \mathbb{V}_2 \prec \mathbb{V}_3$.
1. $\mathbb{N}_{j+1}$ is an end extension of $\mathbb{N}_j$ $(j = 0, 1, 2)$.
2. For $j' > j$, Countable Idealization holds from $\mathbb{V}_j$ to $\mathbb{V}_{j'}$: Let ϕ be an $\in$–formula with parameters from $\mathbb{V}_{j'}$. Then

$$\forall n \in \mathbb{N}_j\, \exists x\, \forall m \in \mathbb{N}\, (m \leq n \;\rightarrow \phi(m, x)) \;\leftrightarrow\; \exists x\, \forall n \in \mathbb{N}_j\, \phi(n, x).$$

3. There is an elementary embedding i_* of $(\mathbb{V}_2;\mathbb{R}_0,\mathbb{R}_1)$ to $(\mathbb{V}_3;\mathbb{R}_1,\mathbb{R}_2)$.
4. There is an elementary embedding i_1 of $(\mathbb{V}_1,\mathbb{R}_0)$ to $(\mathbb{V}_2,\mathbb{R}_1)$ such that $i_1 \restriction \mathbb{N}_0$ is an identity map and $i_1(a) \in \mathbb{N}_2 \setminus \mathbb{N}_1$ for each $a \in \mathbb{N}_1 \setminus \mathbb{N}_0$.
5. There is an elementary embedding i_2 of $\mathbb{V}_2$ to $\mathbb{V}_3$ such that $i_2 \restriction \mathbb{N}_1$ is an identity map and $i_2(a) \in \mathbb{N}_3 \setminus \mathbb{N}_2$ for each $a \in \mathbb{N}_2 \setminus \mathbb{N}_1$.

These requirements are listed as Property 2.1 in arXiv versions v1, v2 of Jin's paper, and appear in a slightly different form in Section 2 of the Discrete Analysis version; see especially Property 2,7 there. Our formulations differ from his in two significant ways.

- Jin works model-theoretically and his universes are superstructures, that is, sets of **ZFC**. In contrast, our universes are proper classes. Nonstandard arguments work similarly in both frameworks.
- In Property 2 Jin postulates Countable Saturation, while the weaker Countable Idealization stated here is more suited for the axiomatic approach. In all instances where 2 is used in Jin's proof, Countable Idealization suffices.

Proposition 6.1 **SPOTS** *interprets Jin's Properties 0. – 5.*

Proof We define: $\mathbb{V}_0 = \mathbb{S}_0$, $\mathbb{V}_1 = \mathbb{S}_{\{0\}}$, $\mathbb{V}_2 = \mathbb{S}_{\{0,1\}}$, $\mathbb{V}_3 = \mathbb{S}_{\{0,1,2\}}$, $i_1 = \mathbf{I}^{\{1\}}_{\{0\}}$, $i_2 = \mathbf{I}^{\{0,2\}}_{\{0,1\}}$ and $i_* = \mathbf{I}^{\{1,2\}}_{\{0,1\}}$.

Property 0. This follows from **GT**, and Property 1. from Proposition 4.4.

Property 2. Countable Idealization is a consequence of **SPOT**, so it suffices to show that each $(\mathbb{S}_{j'},\in,\mathbb{S}_j)$ satisfies the axioms of **SPOT**. The axiom **SP** is the only issue.

SP holds in $(\mathbb{I},\in,\mathbb{S}_0)$, hence it holds in every $(\mathbb{I},\in,\mathbb{S}_j)$ by **HO**. Its validity in $(\mathbb{S}_{j'},\in,\mathbb{S}_j)$ follows.

Property 3. If $\psi(v_1,\dots,v_r)$ is a formula in the common language of the structures $(\mathbb{S}_2,\in,\mathbb{S}_0,\mathbb{S}_1)$ and $(\mathbb{S}_3,\in,\mathbb{S}_1,\mathbb{S}_2)$, then, by **HO**,

$$\forall x_1,\dots,x_r \in \mathbb{S}_2\ [\psi^{\mathbb{S}_2}(x_1,\dots,x_r) \leftrightarrow \psi^{\mathbb{S}_3}(\mathbf{I}^{\{1,2\}}_{\{0,1\}}(x_1),\dots,\mathbf{I}^{\{1,2\}}_{\{0,1\}}(x_r))].$$

Properties 4. and 5. These follow from Propositions 4.3 and 4.4 and the observation that $i_1 = i_* \restriction \mathbb{V}_1$. □

It remains to show that **SPOT** proves the existence of densities used by Jin. This requires a careful appeal to Standardization.

Definition 6.2 In our notation:

(1) For finite $A \subseteq \mathbb{N}$ with $|A|$ unlimited, the *strong upper Banach density of* A is defined by:

$$SD(A) = \sup{}^{\mathbf{st}}\{\mathbf{sh}(|A \cap P|/|P|) \mid |P| \text{ is unlimited}\}$$

(2) If $S \subseteq \mathbb{N}$ has $SD(S) = \eta \in \mathbb{R}$ (note η is standard) and $A \subseteq S$, the *strong upper Banach density of* A *relative to* S is defined by:

$$SD_S(A) = \sup{}^{\mathbf{st}}\{\mathbf{sh}(|A \cap P|/|P|) \mid |P| \text{ is unlimited} \wedge \mathbf{sh}(|S \cap P|/|P|) = \eta\}$$

SPOT does not prove the existence of the standard sets of reals whose supremum needs to be taken (it does not allow Standardization over the uncountable set $\mathbb{R}$), but for the purpose of obtaining the supremum, a set of reals can be replaced by a set of rationals.

Proposition 6.3 **SPOT** *proves the existence of* $SD_S(A)$.

Proof We note that $SD_S(A) = \sup{}^{\mathbf{st}}\{q \in \mathbb{Q} \mid \Phi(q)\}$ where $\Phi(q)$ is the formula:

$$\exists P\,[\,\forall^{st}_{\mathbb{N}} i\,(|P| > i) \wedge \forall^{st}_{\mathbb{N}} j(|\,|S \cap P|/|P| - \eta| < \tfrac{1}{j+1}) \wedge q \leq |A \cap P|/|P|\,]$$

The formula Φ is equivalent to

$$\exists P\,\forall^{st}_{\mathbb{N}} i\,[\,(|P| > i) \wedge (|\,|S \cap P|/|P| - \eta| < \tfrac{1}{i+1}) \wedge q \leq |A \cap P|/|P|\,]$$

which, upon the exchange of the order of $\exists P$ and $\forall^{st}_{\mathbb{N}} i$, enabled by Countable Idealization, converts to a special $\mathbf{st}$-$\in$–formula:

$$\forall^{st}_{\mathbb{N}} i\,\exists P\,[\,(|P| > i) \wedge (|\,|S \cap P|/|P| - \eta| < \tfrac{1}{i+1}) \wedge q \leq |A \cap P|/|P|\,]$$

Proposition 2.4 concludes the proof. □

The definitions of these densities relativize to every level $j > 0$. Their existence at higher levels in **SPOTS** follows from Proposition 4.5.

7 Analysis with ultrasmall numbers

The presentation of analysis in Hrbacek, Lessmann and O'Donovan [14] is based on the notion of *relative observability*, which we denote by $\sqsubseteq$. In this section, $\mathbf{S}_x$ is the class $\{y \mid y \sqsubseteq x\}$ and, more generally, $y \in \mathbf{S}_{x_1,\ldots,x_k}$ means that $y \sqsubseteq x_i$ for some $1 \leq i \leq k$. The elements of $\mathbf{S}_\emptyset$ are *always observable* (= standard). We write formulas using

this class notation. Let $\Phi(v_1, \ldots . v_k)$ be a **st**-$\in$–formula; then $\Phi(v_1, \ldots . v_k; \mathbf{S})$ is the formula obtained from Φ by replacing each occurrence of $\mathbf{st}(v)$ with $v \in \mathbf{S}$. We use $\bar{x}$ as shorthand for the list $x_1, \ldots, x_k$.

The following principles are postulated in [14] (see the Appendix, especially pages 277–281).

Relative Observability Principle: For all x, y, z

(1) $x \sqsubseteq x$
(2) If $x \sqsubseteq y$ and $y \sqsubseteq z$, then $x \sqsubseteq z$
(3) If not $x \sqsubseteq y$, then $y \sqsubseteq x$
(4) $0 \sqsubseteq x$
(5) $\forall x \exists y \, (x \sqsubseteq y \wedge y \not\sqsubseteq x)$

Existence Principle: There exist $h \in \mathbb{R}$ such that $h \simeq 0$, $h \neq 0$.

Observable Neighbor Principle: $\forall x \in \mathbb{R} \, (x \text{ limited } \rightarrow \exists r \in \mathbb{R} \cap \mathbf{S}_{\emptyset} \, (x \simeq r))$.

Stability Principle: Assume the variables p, q do not appear in Φ.

$$p \sqsubseteq q \rightarrow \forall \bar{x} \in \mathbf{S}_p \, (\Phi(\bar{x}; \mathbf{S}_p) \leftrightarrow \Phi(\bar{x}; \mathbf{S}_q)).$$

Definition 7.1 Formulas of the form $\Phi(\bar{x}; \mathbf{S}_{\bar{x}})$ are *internal formulas.* (We assume that $x_1, \ldots . x_k$ do not appear as bound variables in Φ.)

Stability for internal formulas can be restated as follows (let $p = \langle \bar{x} \rangle$, $q = \langle \bar{x}, \bar{y} \rangle$):

$$\forall \bar{x}, \bar{y} \, (\Phi(\bar{x}; \mathbf{S}_{\bar{x}}) \leftrightarrow \Phi(\bar{x}; \mathbf{S}_{\bar{x},\bar{y}}))$$

Closure Principle: Let $\Phi(x, \bar{x}; \mathbf{S}_{x,\bar{x}})$ be an internal formula.

$$\exists x \, \Phi(x, \bar{x}; \mathbf{S}_{x,\bar{x}}) \rightarrow \exists x \in \mathbf{S}_{\bar{x}} \, \Phi(x, \bar{x}; \mathbf{S}_{\bar{x}}).$$

Definition Principle Let $\Phi(x, \bar{x}; \mathbf{S}_{x,\bar{x}})$ be an internal formula. For every set A and all $x_1, \ldots, x_k$ there is a set $B \in \mathbf{S}_{A,\bar{x}}$ such that

$$\forall x \, (x \in B \leftrightarrow x \in A \wedge \Phi(x, \bar{x}; \mathbf{S}_{x,\bar{x}})).$$

Let **HLOD** be the theory in the $\sqsubseteq$-$\in$–language whose axioms are **ZF** plus the above principles. In the rest of this section we show that **HLOD** can be interpreted in **SPOTS**. In combination with Theorem 4.7 this shows that the presentation of analysis in [14] relies at the most on the Axiom of Countable Choice, as is customary in traditional analysis.

In the rest of this section we work in **SPOTS**. We recall that n, m range over standard finite natural numbers and define:

$$x \sqsubseteq y \;\leftrightarrow\; \forall \mathsf{n}\, (y \in \mathbb{S}_{\mathsf{n}} \to x \in \mathbb{S}_{\mathsf{n}})$$

By the axiom **SF**, for every x there is a least standard $n \in \mathbb{N}$ such that $x \in \mathbb{S}_n$; we denote it $\mathsf{n}(x)$ (the *level of* x). In this notation, $x \sqsubseteq y \;\leftrightarrow\; \mathsf{n}(x) \le \mathsf{n}(y)$ and $\mathbf{S}_x = \mathbb{S}_{\mathsf{n}(x)}$.

Validity of the Relative Observability Principle in this interpretation is trivial, and Existence and Observable Neighbor follow immediately from the analogous principles of **SPOTS**. Stability Principle is Proposition 4.5. We also note that $\mathbf{S}_{x_1,\dots,x_k} = \mathbf{S}_{\langle x_1,\dots,x_k\rangle} = \mathbb{S}_{\mathsf{n}(\langle x_1,\dots,x_k\rangle)}$.

To prove Closure, assume $\exists x\, \Phi(x, \bar{x}; \mathbf{S}_{x,\bar{x}})$. Fix some x such that $\Phi(x, \bar{x}; \mathbf{S}_{x,\bar{x}})$ holds, and let $p = \langle \bar{x} \rangle$, $q = \langle x, \bar{x} \rangle$. From the formula $\exists x \in \mathbf{S}_q\ \Phi(x, \bar{x}; \mathbf{S}_q)$ we get $\exists x \in \mathbf{S}_p\ \Phi(x, \bar{x}; \mathbf{S}_p)$ by the Stability Principle.

It remains to prove the Definition Principle. Let $\Phi(x, \bar{x}; \mathbf{S}_{x,\bar{x}})$ be an internal formula. By **SN** we get

$$\forall \bar{x} \in \mathbb{S}_0\ \forall A \in \mathbb{S}_0\ \exists B \in \mathbb{S}_0\ \forall x \in \mathbb{S}_0\ (x \in B \;\leftrightarrow\; x \in A \,\wedge\, \Phi(x, \bar{x}; \mathbb{S}_0)).$$

By applying Proposition 4.5 to this statement we get, for any n,

$$\forall \bar{x} \in \mathbb{S}_{\mathsf{n}}\ \forall A \in \mathbb{S}_{\mathsf{n}}\ \exists B \in \mathbb{S}_{\mathsf{n}}\ \forall x \in \mathbb{S}_{\mathsf{n}}\ (x \in B \;\leftrightarrow\; x \in A \,\wedge\, \Phi(x, \bar{x}; \mathbb{S}_{\mathsf{n}})).$$

We now fix $\bar{x}$ and A, let $\mathsf{n} = \mathsf{n}(\langle A, \bar{x} \rangle)$, and let $B \in \mathbb{S}_{\mathsf{n}}$ be such that

$$\forall x \in \mathbb{S}_{\mathsf{n}}\ (x \in B \;\leftrightarrow\; x \in A \,\wedge\, \Phi(x, \bar{x}; \mathbb{S}_{\mathsf{n}})).$$

Applying Proposition 4.5 to this formula (with parameters $A, B, \bar{x} \in \mathbb{S}_{\mathsf{n}}$) we obtain that for any $\mathsf{m} \ge \mathsf{n}$,

$$\forall x \in \mathbb{S}_{\mathsf{m}}\ (x \in B \;\leftrightarrow\; x \in A \,\wedge\, \Phi(x, \bar{x}; \mathbb{S}_{\mathsf{m}})).$$

For arbitrary x take $\mathsf{m} = \mathsf{n}(\langle A, x, \bar{x} \rangle)$ to get

$$x \in B \;\leftrightarrow\; x \in A \,\wedge\, \Phi(x, \bar{x}; \mathbb{S}_{\mathsf{m}}) \;\leftrightarrow\; x \in A \,\wedge\, \Phi(x, \bar{x}; \mathbf{S}_{A,x,\bar{x}}).$$

It remains to notice that $\Phi(x, \bar{x}; \mathbf{S}_{A,x,\bar{x}}) \;\leftrightarrow\; \Phi(x, \bar{x}; \mathbf{S}_{x,\bar{x}})$, by Stability.

Remark 1 In [14] the basic concepts of calculus, such as continuity, limit, derivative and integral, are defined by internal formulas involving ultrasmall numbers (infinitesimals). It is then necessary to be able to include such previously defined internal concepts in subsequent internal formulas. This move can be justified in several different ways.

- One can prove the equivalence of the nonstandard definitions of these concepts

with the traditional ε-δ–definitions. This requires only straightforward, familiar arguments, but it is somewhat against the spirit of nonstandard approach.

- The presentation in [14] relies on a general result: *Every internal formula* $\Phi(\bar{x}; \mathbf{S}_{\bar{x}})$ *is equivalent to an* $\in$*–formula* $\phi(\bar{x})$. This is a consequence of the Reduction Theorem (Kanovei and Reeken [23, Theorem 3.2.3]). However, the proof of Reduction Theorem uses a strong form of **AC** (Boolean Prime Ideal Theorem). It is not clear whether some version of it could be proved in **HLOD** when **AC** is not available.
- We can avoid **AC** by relying instead on the Definition Principle. Let $R(\bar{z})$ be a predicate defined by an internal formula $\Psi(\bar{z}, \mathbf{S}_{\bar{z}})$ and let A be a standard set such that $\forall \bar{z}\,(\Psi(\bar{z}, \mathbf{S}_{\bar{z}}) \to \bar{z} \in A)$. The Definition Principle (with $\bar{z}$ in place of x and empty list $\bar{x}$) provides a standard set B such that

$$\forall \bar{z}\,(\langle \bar{z} \rangle \in B \;\leftrightarrow\; \langle \bar{z} \rangle \in A \wedge \Psi(\bar{z}, \mathbf{S}_{\bar{z}}) \;\leftrightarrow\; \Psi(\bar{z}, \mathbf{S}_{\bar{z}}) \;\leftrightarrow\; R(\bar{z})).$$

  If $\Phi(\bar{x}, \mathbf{S}_{\bar{x}})$ is a formula where the predicate $R(\bar{z})$ also occurs, perhaps with some of the variables $\bar{z}$ quantified (a *generalized internal formula*), we can replace each such occurence by its equivalent $\langle \bar{z} \rangle \in B$ and convert Φ to an internal formula as in Definition 7.1.

  Here is one example. The derivative of a real-valued function f at $a \in \mathbb{R}$ is defined in terms of infinitesimals by an internal formula at the beginning of Section 4. Let $\mathbb{F} = \{f \subseteq \mathbb{R} \times \mathbb{R} \mid f \text{ is a function}\}$ and let $A = \mathbb{F} \times \mathbb{R} \times \mathbb{R}$. The Definition Principle guarantees the existence of a standard set $B = \{\langle f, a, b \rangle \in A \mid f'(a) = b\}$. Any generalized internal formula involving the notion of derivative can in principle be converted to an internal formula by replacing each occurence of $f'(a) = b$ with $\langle f, a, b \rangle \in B$. In practice there is hardly ever any need to carry out such conversions; it suffices to keep in mind that Stability, Closure and Definition Principles apply to generalized internal formulas.

Remark 2 The proof of the local Peano Theorem in [14, Theorems 125 and 161] uses Standardization over $\mathbb{R}$, which implies the existence of nonprincipal ultrafilters over $\mathbb{N}$. The density of $\mathbb{Q}$ in $\mathbb{R}$ can be used to replace the argument by one that uses only Countable Standardization, which is available in **SCOTS**. Actually, Countable Standardization for $\mathbb{N}$–special formulas, available in **SPOTS**, suffices. See Hrbacek and Katz [17] for details; a similar idea is also used in the proof of the existence of Banach densities in Proposition 6.3 of this paper.

8 Conservativity

Conservativity of **SPOT** over **ZF** was established in Hrbacek and Katz [15] by a construction that extends and combines the methods of forcing developed by Ali Enayat [8] and Mitchell Spector [34]. Conservativity of **SCOT** over **ZF** + **ADC** is obtained there as a corollary. Here we give a simple, more direct proof of the latter result that generalizes straightforwardly to the proof of conservativity of **SCOTS** over **ZF** + **ADC**.

We prove the following proposition.

Proposition 8.1 *Every countable model* $\mathcal{M} = (M, \in^{\mathcal{M}})$ *of* **ZF** + **ADC** *has an extension to a model of* **SCOTS** *in which elements of* M *are exactly the standard sets.*

The difficulty is that $\mathcal{M}$ may contain no nonprincipal ultrafilters. We add such an ultrafilter to $\mathcal{M}$ by forcing, and then carry out the construction of the iterated ultrapower as in Section 3 inside this generic extension of $\mathcal{M}$.

Jech [19] is the standard reference for forcing and generic extensions of well-founded models of **ZF**. For details on the extension of this material to non-well-founded models see Corazza [6, 7].

8.1 Forcing

In this subsection we work in **ZF** + **ADC**.

Definition 8.2 Let $\mathbb{P} = \{p \subseteq \omega \mid p \text{ is infinite}\}$. For $p, p' \in \mathbb{P}$ we say that p' *extends* p (notation: $p' \leq p$) if $p' \subseteq p$.

The poset $\mathbb{P}$ is not *separative* (Jech [19, Section 17]); forcing with $\mathbb{P}$ is equivalent to forcing with $\widetilde{\mathbb{P}} = \mathcal{P}^{\infty}(\omega)/_{\mathbf{fin}}$.

The poset $\widetilde{\mathbb{P}}$ is ω*–closed*: if $\langle p_n \mid n \in \omega \rangle$ is a sequence of conditions from $\mathbb{P}$ such that, for each $n \in \omega$, $p_{n+1} \setminus p_n$ is finite, then there is $p \in \mathbb{P}$ such that $p \setminus p_n$ is finite for all $n \in \omega$. It follows that the forcing with $\mathbb{P}$ does not add any new countable sets (note that the proof of this fact uses **ADC**).

The forcing notion $\mathbb{P}$ is *homogeneous* in the sense that for $x_1, \ldots, x_s \in \mathbb{V}$ and $p, p' \in \mathbb{P}$ we have $p \Vdash \phi(\check{x}_1, \ldots, \check{x}_s)$ if and only if $p' \Vdash \phi(\check{x}_1, \ldots, \check{x}_s)$. (Jech [19, Lemma 19.10 and related material].)

This is a consequence of the following fact (we let $p^c = \mathbb{N} \setminus p$). For all $p_1, p_2 \in \mathbb{P}$ such that p_1^c, p_2^c are infinite, there is an automorphism π of $\mathbb{P}$ such that $\pi(p_1) = p_2$. It can be obtained as follows: fix a one-one mapping α of ω onto ω such that α maps p_1 onto p_2 in an order-preserving way, and maps p_1^c onto p_2^c in an order-preserving way, and then define $\pi(p) = \alpha[p]$.

8.2 Generic Extensions

Let $\mathcal{M} = (M, \in^{\mathcal{M}})$ be a countable model of **ZF** + **ADC** and let $\mathcal{G}$ be an $\mathcal{M}$–generic filter over $\mathbb{P}^{\mathcal{M}}$. The generic extension $\mathcal{M}[\mathcal{G}]$ is a model of **ZF** + **ADC** extending $\mathcal{M}$ and the forcing does not add any new reals or countable subsets of M, ie, every countable subset of M in $\mathcal{M}[\mathcal{G}]$ belongs to M.

We need the following observation. The structure $(M[\mathcal{G}], \in^{\mathcal{M}[\mathcal{G}]}, M)$ is a model of $(\mathbf{ZF} + \mathbf{ADC})^{\mathbb{M}}$, a theory obtained by adding a unary predicate symbol $\mathbb{M}$ to the $\in$–language of **ZF** and postulating that the axioms of Separation, Replacement and Dependent Choice hold for formulas in this extended language. This is a piece of folklore; a proof can be given by adding the predicate $\mathbb{M}$ to the forcing language and defining

$$p \Vdash \mathbb{M}(x) \iff \forall p' \leq p \, \exists p'' \leq p' \, \exists z \, (p'' \Vdash x = \check{z}).$$

One can then prove the appropriate versions of Forcing Theorem and the Generic Model Theorem as in Jech [19, Section 18].

8.3 Conservativity of SCOTS over ZF + ADC.

We work in the structure $(M[\mathcal{G}], \in^{\mathcal{M}[\mathcal{G}]}, M)$, a model of $(\mathbf{ZF} + \mathbf{ADC})^{\mathbb{M}}$, and use ω to denote its set of natural numbers. The generic filter $\mathcal{G}$ is a nonprincipal ultrafilter over ω and one can construct the expanded iterated ultrapower

$$\mathcal{M}_\infty = (\mathbb{M}_\infty, \in_\infty, \mathbb{M}_{\mathsf{a}}, \Pi^{\mathsf{b}}_{\mathsf{a}}; \mathsf{a}, \mathsf{b} \in \mathcal{P}^{\mathbf{fin}}(\omega), |\mathsf{a}| = |\mathsf{b}|)$$

of $\mathbb{M}$ by $\mathcal{G}$ as in Section 3 (let $I = \omega$, $U = \mathcal{G}$, and replace $\mathbb{V}$ by $\mathbb{M}$).

Łoś's Theorem holds because **ACC** is available, and Π^∞_0 canonically embeds $\mathbb{M}$ into $(\mathbb{M}_\infty, \in_\infty)$. The structure $\mathcal{M}_\infty$ interprets the language of **SPOTS** (with $\mathbb{S}_{\mathsf{a}}$ interpreted as $\mathbb{M}_{\mathsf{a}}$ and $\mathbf{I}^{\mathsf{b}}_{\mathsf{a}}$ interpreted as $\Pi^{\mathsf{b}}_{\mathsf{a}}$). As in Proposition 4.2, the structure $\mathcal{M}_\infty$ satisfies **IS**, **GT** and **HO**. It remains to show that **SN** and **DC** hold there.

Proposition 8.3 **DC** *holds in* $\mathcal{M}_\infty$.

Proof Let $\Phi(u,v,w)$ be a formula in the language of **SPOTS**. Let $b \in \mathbb{M}_{\mathsf{a}}$ and $c \in \mathbb{M}_\infty$ be such that

$$\Psi(B,b,c): \qquad [b \in \mathbb{S}_{\mathsf{a}} \wedge \forall x \in \mathbb{S}_{\mathsf{a}}\ \exists y \in \mathbb{S}_{\mathsf{a}}\ \Phi(x,y,c)]^{\mathcal{M}_\infty}$$

holds. (The superscript $\mathcal{M}_\infty$ indicates that the quantifiers range over $\mathbb{M}_\infty$ and all symbols are interpreted in $\mathcal{M}_\infty$.) Ψ is (equivalent to) a formula of the forcing language (we identify b, c and a with their names in the forcing language), hence there is $p \in \mathbb{P} \cap \mathcal{G}$ such that $p \Vdash \Psi$. Let $p_0 \leq p$.

We let the variable a (with decorations) range over the names in the forcing language and define the class:

$$\mathbf{A} = \{\langle p', a' \rangle \mid p' \leq p_0 \wedge p' \Vdash [a' \in \mathbb{S}_{\mathsf{a}}]^{\mathcal{M}_\infty}\}$$

Note that $\langle p_0, b\rangle \in \mathbf{A}$, and define $\mathbf{R}$ on $\mathbf{A}$ by:

$$\langle p', a'\rangle \mathbf{R} \langle p'', a''\rangle \text{ if and only if } p'' \leq p' \wedge p'' \Vdash \Phi^{\mathcal{M}_\infty}(a', a'', c)$$

It is clear from the properties of forcing that for every $\langle p', a'\rangle \in \mathbf{A}$ there is $\langle p'', a''\rangle \in \mathbf{A}$ such that $\langle p', a'\rangle \mathbf{R} \langle p'', a''\rangle$. Using Reflection and **ADC** we obtain a sequence $\langle \langle p_n, a_n\rangle \mid n \in \omega\rangle$ such that $a_0 = b$, and, for all $n \in \omega$, $\langle p_n, a_n\rangle \in A$, $p_{n+1} \leq p_n$ and $p_{n+1} \Vdash \Phi^{\mathcal{M}_\infty}(a_n, a_{n+1}, c)$.

As the forcing is ω–closed, one obtains $p_\infty \in \mathbb{P}$ and $\langle a_n \mid n \in \omega\rangle$ such that $p_\infty \leq p_0$ and $p_\infty \Vdash [a_n \in \mathbb{S}_{\mathsf{a}} \wedge \Phi(a_n, a_{n+1}, c)]^{\mathcal{M}_\infty}$ for all $n \in \omega$.

By the genericity of $\mathcal{G}$ there is some $p_\infty \in \mathcal{G}$ and the associated sequence $\langle a_n \mid n \in \omega\rangle$ with this property. Hence $(M[\mathcal{G}], \in^{\mathcal{M}[\mathcal{G}]}, M)$ satisfies $[a_n \in \mathbb{S}_{\mathsf{a}} \wedge a_0 = b \wedge \Phi(a_n, a_{n+1}, c)]^{\mathcal{M}_\infty}$ for all $n \in \omega$.

The class $\mathbb{S}_{\mathsf{a}}$ is interpreted in $\mathcal{M}_\infty$ by the ultrapower $\mathbb{M}_{\mathsf{a}} = \mathbb{M}^{I_{\mathsf{a}}}/U_{\mathsf{a}}$ (for $U = \mathcal{G}$) . Since this ultrapower is ω_1–saturated, there is $\bar{b} \in \mathbb{M}_{\mathsf{a}}$ such that

$$[\bar{b} \text{ is a function } \wedge\ \operatorname{dom} \bar{b} = \mathbb{N} \wedge b_n = a_n]^{\mathbb{M}_{\mathsf{a}}}$$

holds for every $n \in \omega$. This translates to the desired

$$[\bar{b} \in \mathbb{S}_{\mathsf{a}} \wedge \operatorname{dom} \bar{b} = \mathbb{N} \wedge b_0 = b \wedge \forall n \in \mathbb{N} \cap \mathbb{S}_0\ \Phi(b_n, b_{n+1}, c)]^{\mathcal{M}_\infty}. \qquad \square$$

Proposition 8.4 **SN** *holds in* $\mathcal{M}_\infty$.

Proof Let $\Phi(u)$ be a formula in the language of **SPOTS** (with no parameters) and $A \in \mathbb{M}$. Let $\Psi(u)$ be the formula $\Phi(u)^{\mathcal{M}_\infty}$ of the forcing language. By homogeneity of

the forcing, $p \Vdash \Psi(\check{a})$ if and only if $p' \Vdash \Psi(\check{a})$ holds for all $a \in A$ and $p, p' \in \mathbb{P}$. Fix some $p \in \mathcal{G}$ and let $S = \{a \in A \mid p \Vdash \Psi(\check{a})\}$. For $a \in \mathbb{S}$ then $a \in S$ if and only if $a \in A \wedge \Phi(a)^{\mathcal{M}_\infty}$ holds. □

The structure $\mathcal{M}_\infty$ is a class model of **SCOTS** constructed inside the countable model $(M[\mathcal{G}], \in^{\mathcal{M}[\mathcal{G}]}, M)$. It converts into a countable model $\widetilde{\mathcal{M}}_\infty$ in the meta-theory so that $\Phi^{\mathcal{M}_\infty} \leftrightarrow \widetilde{\mathcal{M}}_\infty \vDash \Phi$ for all formulas in the language of **SCOTS**.

8.4 Finitistic proofs

The model-theoretic proof of Proposition 8.1 in Subsections 8.1–8.3 is carried out in **ZF**. Using techniques from Simpson [32, Chapter II, especially II.3 and II.8], it can be verified that the proof goes through in $\mathbf{RCA}_0$ (without loss of generality one can assume that $M \subseteq \omega$).

The proof of Theorem 4.6 from Proposition 8.1 requires the Gödel's Completeness Theorem and therefore $\mathbf{WKL}_0$; see [32, Theorem IV.3.3]. We conclude that Theorem 4.6 can be proved in $\mathbf{WKL}_0$.

Theorem 4.6, when viewed as an arithmetical statement resulting from identifying formulas with their Gödel numbers, is Π^0_2. It is well-known that $\mathbf{WKL}_0$ is conservative over **PRA** (Primitive Recursive Arithmetic) for Π^0_2 sentences ([32, Theorem IX.3.16)]; therefore Theorem 4.6 is provable in **PRA**. The theory **PRA** is generally considered to correctly capture finitistic reasoning (see eg Simpson [32, Remark IX.3.18]). We conclude that Theorem 4.6 has a finitistic proof.

These remarks apply equally to Theorem 4.7 and Proposition 8.7.

8.5 Conservativity of SPOTS over ZF + ACC.

The proof of conservativity of **SCOTS** over **ZF**+**ADC** presented in Subsections 8.1–8.3 relies on **ADC** in three places.

(1) To prove that forcing with $\mathbb{P}$ does not add new countable sets
(2) To prove that Łoś's Theorem (Proposition 3.1) holds in $\mathcal{M}_\infty$
(3) To prove that **DC** holds in $\mathcal{M}_\infty$ (Proposition 8.3)

Łoś's Theorem (2) requires only **ACC**. We establish weaker versions of (1) and (3) assuming only **ZF**.

Let $\mathcal{M} = (M, \in^{\mathcal{M}})$ be a countable model of **ZF** and let $\mathcal{G}$ be a generic filter over $\mathbb{P}^{\mathcal{M}}$. We work in the structure $(M[\mathcal{G}], \in^{\mathcal{M}[\mathcal{G}]}, M)$, a model of $\mathbf{ZF}^{\mathbb{M}}$, and use ω to denote its set of natural numbers.

Proposition 8.5 (1) *The generic filter $\mathcal{G}$ is an $\mathcal{M}$-*ultrafilter*:*

$$\text{If } n \in \omega, \langle A_i \rangle_{i \in n} \in M \text{ and } \bigcup_{i \in n} A_i \in \mathcal{G}, \text{ then } A_i \in \mathcal{G} \text{ for some } i \in n.$$

(2) *The generic filter $\mathcal{G}$ is $\mathcal{M}$-*iterable*:*

$$\text{If } S \in M \text{ and } S \subseteq \omega \times \omega, \text{ then } \{i \in \omega \mid \{j \in \omega \mid \langle i,j \rangle \in S\} \in \mathcal{G}\} \in M.$$

Proof (See Enayat [8].)

(1) For every $p \in \mathbb{P}$, $p \subseteq \bigcup_{i \in n} A_i \in \mathcal{G}$, there is $p' \leq p$ such that $p' \subseteq A_i$ for some $i \in n$.

(2) Let $S_i^0 = \{j \in \omega \mid \langle i,j \rangle \in S\}$ and $S_i^1 = \omega \setminus S_i^0$.

We say that $p \in \mathbb{P}$ *decides* S_i^0 if either $p \setminus S_i^0$ or $p \setminus S_i^1$ is bounded. We prove that for every p there is $p^* \in \mathbb{P}$ such that $p^* \leq p$ and p^* decides S_i^0 for all $i \in \omega$. It then follows that some such p^* is in $\mathcal{G}$, and $\{i \in \omega \mid S_i^0 \in \mathcal{G}\} = \{i \in \omega \mid p^* \setminus S_i^0 \text{ is bounded}\} \in M$.

For $t \in 2^n$ we let $|t| = n$ and $S_t = \bigcap_{i \in |t|} S_i^{t_i}$ ($S_\emptyset = \omega$). We define a tree $T \subseteq 2^{<\omega}$ by $t \in T$ if and only if $p \cap S_t$ is unbounded. Since $\bigcup_{|t|=n} S_t = \omega$, the tree T is infinite. By König's Lemma T has an infinite branch t^*. We let $p_i = p \cap S_{t^* \restriction i} \in \mathbb{P}$; clearly $p_0 = p$ and $p_{i+1} \subseteq p_i$ for all $i \in \omega$. We let $n_0 =$ the least element of p_0 and $n_{i+1} =$ the least element of p_{i+1} greater than n_i. Then $p^* = \{n_i \mid i \in \omega\}$ is as required. □

This proposition enables the recursive definition of tensor powers $\otimes^n \mathcal{G}$ and an inductive proof that, for $S \in M$, $S \subseteq \omega \times \omega^n$, we have $\{i \in \omega \mid \{\mathbf{j} \in \omega^n \mid \langle i, \mathbf{j} \rangle \in S\} \in \otimes^n \mathcal{G}\} \in M$. The expanded iterated ultrapower $\mathcal{M}_\infty$ for $I = \omega$ and $U = \mathcal{G}$ is defined as in Section 3, with the understanding that $\mathbb{V}$ is replaced by $\mathbb{M}$ and only functions in $\mathbb{M}$ are employed; ie., $\mathbb{V}^I$ is replaced by $\mathbb{M}^I \cap \mathbb{M}$ everywhere. In particular, $\mathbb{V}_{\mathsf{a}} = \mathbb{V}^{I_{\mathsf{a}}}/U_{\mathsf{a}}$ is replaced by $\mathbb{M}_{\mathsf{a}} = (\mathbb{M}^{I_{\mathsf{a}}} \cap \mathbb{M})/U_{\mathsf{a}}$ (the definition of $[f]_{U_{\mathsf{a}}}$ is also restricted to $g, h \in \mathbb{M}$).

If we assume that $\mathcal{M}$ satisfies **ACC**, Łoś's Theorem holds and the structure $\mathcal{M}_\infty$ satisfies **IS**, **GT**, **HO** and **SN**. It remains to show that the Standard Part axiom holds there.

Proposition 8.6 **SP** *holds in $\mathcal{M}_\infty$.*

Proof Let $F \in M$, $F\colon\ \omega \to \mathcal{P}(\omega)$ (so $[F]_{\mathcal{G}}$ is a subset of $[c_\omega]_{\mathcal{G}}$ in $(M^\omega \cap M)/\mathcal{G}$). Define $S \in M$ by $\langle i,j\rangle \in S$ if and only if $i \in F(j)$. $\mathcal{M}$–iterability of $\mathcal{G}$ implies that $B = \{i \in \omega \mid \{j \in \omega \mid \langle i,j\rangle \in S\} \in \mathcal{G}\} \in M$. Now:

$$[c_i]_{\mathcal{G}} \in_{\mathcal{G}} [c_B]_{\mathcal{G}} \ \leftrightarrow\ i \in B \ \leftrightarrow\ \{j \in \mathbb{N} \mid i \in F(j)\} \in \mathcal{G} \ \leftrightarrow\ [c_j]_{\mathcal{G}} \in_{\mathcal{G}} [F]_{\mathcal{G}} \qquad \square$$

We conclude that $\mathcal{M}_\infty$ is a model of **SPOTS**.

8.6 Conservativity of SPOT$^+$ over ZF.

The forcing construction used to establish conservativity of **SPOT**$^+$ over **ZF** is much more complicated because one needs to force both a generic filter $\mathcal{G}$ and the validity of Łoś's Theorem in the corresponding "extended ultrapower." We describe the appropriate forcing conditions (see [15]).

Let $\mathbb{Q} = \{q \in \mathbb{V}^\omega \mid \exists k \in \omega\ \forall i \in \omega\,(q(i) \subseteq \mathbb{V}^k \wedge q(i) \neq \emptyset)\}$.

The number k is the *rank* of q. We note that $q(i)$ for each $i \in \omega$, and q itself, are sets, but $\mathbb{Q}$ is a proper class.

The forcing notion $\mathbb{H}$ is defined as follows: $\mathbb{H} = \mathbb{P} \times \mathbb{Q}$ and $\langle p', q'\rangle \in \mathbb{H}$ *extends* $\langle p, q\rangle \in \mathbb{H}$ if and only if p' extends p, $\operatorname{rank} q' = k' \geq k = \operatorname{rank} q$, and for almost all $i \in p'$ and all $\langle x_0, \ldots, x_{k'-1}\rangle \in q'(i)$, $\langle x_0, \ldots, x_{k-1}\rangle \in q(i)$.

The forcing with $\mathbb{H}$ adds many new reals; in fact, it makes all ordinals countable.

Proposition 8.7 **SPOT**$^+$ *is a conservative extension of* **ZF**.

Proof Conservativity of **SPOT** + **SN** over **ZF** is established in [15, Theorem B] via forcing with $\mathbb{H}$. It remains only to show that **SF** also holds in the model constructed there.

In [15, Definition 4.4] forcing is defined for $\in$–formulas only, but the definition can be extended to **st**-$\in$–formulas by adding the clause

$$\langle p, q\rangle \Vdash \mathbf{st}(\dot{G}_n) \text{ if and only if } \operatorname{rank} q = k > n \text{ and } \exists x\ \forall^{\mathbf{aa}} i \in p\ \forall \langle x_0, \ldots, x_{k-1}\rangle \in q(i)\ (x_n = x). \tag{11}$$

[15, Proposition 4.6 ("Łoś's Theorem")] does not hold for **st**-$\in$–formulas, but the equivalence of clauses (1) and (2) in [15, Proposition 4.12 (The Fundamental Theorem of Extended Ultrapowers)] remains valid ($\mathfrak{N}$ is the extended ultrapower of $\mathcal{M}$):

Let $\Phi(v_1, \ldots, v_s)$ *be an* **st**-$\in$*–formula with parameters from* M*. If* $G_{n_1}, \ldots, G_{n_s} \in \mathfrak{N}$*, then the following statements are equivalent:*

(1) $\mathfrak{N} \vDash \Phi(G_{n_1}, \ldots, G_{n_s})$

(2) *There is some* $\langle p, q\rangle \in \mathcal{G}$ *such that* $\langle p, q\rangle \Vdash \Phi(\dot{G}_{n_1}, \ldots, \dot{G}_{n_s})$ *holds in* $\mathcal{M}$.

We now prove that **SF** holds in $\mathfrak{N}$.

Without loss of generality we can assume $A = N \in \omega$. For every $\langle p, q\rangle \in \mathbb{H}$ and every $n \in N$ there is $\langle p'g'\rangle \leq \langle p, q\rangle$ such that $\langle p'g'\rangle$ decides $\Phi(\check{n})$. By induction on N, for every $\langle p, q\rangle \in \mathbb{H}$ there is $\langle p_N, q_N\rangle \leq \langle p, q\rangle$ that decides $\Phi(\check{n})$ for all $n \in N$ simultaneously. Hence there is $\langle \widetilde{p}, \widetilde{q}\rangle \in \mathcal{G}$ with this property. Let $S = \{n \in N \mid \langle \widetilde{p}, \widetilde{q}\rangle \Vdash \Phi(\check{n})\}$. By the Fundamental Theorem, $S = \{n \in N \mid \mathfrak{N} \vDash \Phi(n)\}$. □

Final Remark. Labels a, b in **SPOTS** range over standard finite sets. This implies that the levels of standardness are enumerated by standard natural numbers. It is an open question whether one could allow labels to range over all finite sets, ie, to have levels of standardness indexed by all natural numbers. Theories of this kind have been developed in Hrbacek [12] on the basis of **ZFC**. It seems likely that the present work could be generalized analogously.

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Department of Mathematics, City College of CUNY, New York, NY 10031,

khrbacek@icloud.com

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