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On perturbations of Hilbert spaces and probability algebras with a generic automorphism[1]

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We prove that IHS_A, the theory of infinite dimensional Hilbert spaces equipped with a generic automorphism, is $\aleph_0$-stable up to perturbation of the automorphism, and admits prime models up to perturbation over any set. Similarly, APr_A, the theory of atomless probability algebras equipped with a generic automorphism is $\aleph_0$-stable up to perturbation. However, not allowing perturbation it is not even superstable.

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Introduction

It was proved by Chatzidakis and Pillay [11] that if T is a first order superstable theory, and the theory $T_\tau = T \cup \{\tau \text{ is an automorphism}\}$ has a model companion T_A, then T_A is supersimple. Throughout this paper we refer to T_A (when it exists) as the theory of models of T equipped with a *generic* automorphism.

Continuous first order logic is an extension of first order logic, introduced in [7] as a formalism for a model theoretic treatment of metric structures (see also [6] for a general exposition of the model theory of metric structures). It is a natural question to ask whether the theorem of Chatzidakis and Pillay generalises to continuous logic and metric structures.

The proof of Chatzidakis and Pillay would hold in metric structures if we used the classical definitions of superstability and supersimplicity literally (namely, types do not fork over finite sets). These definitions, however, are known to be too strong in

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metric structures, and need to be weakened somewhat in order to make sense. For example, the theory of Hilbert spaces has a countable language, is totally categorical and does not satisfy the classical definition of superstability.

The standard definition for $\aleph_0$-stability and superstability for metric structures ([14], and later [2]) comes from measuring the size of a type space not by its cardinality but by its density character in the metric induced on it from the structures. A continuous theory is supersimple if for every $\varepsilon > 0$, the ε-neighbourhood of a type does not fork over a finite set of parameters, or equivalently, if ordinal Lascar ranks corresponding to "ε-dividing" exist. A theory is superstable if and only if it is stable and supersimple. Similarly, $\aleph_0$-stability is equivalent to the existence of ordinal ε-Morley ranks, which may be defined via a metric variant of the classical Cantor-Bendixson ranks (see [4] for a general study of such ranks).

With these corrected definitions, the class of $\aleph_0$-stable theories is rich with examples: Hilbert spaces, probability algebras, L^p Banach lattices and so on. Furthermore, many classical results can be generalised. For example, an $\aleph_0$-stable theory has prime models over every set, an uncountably categorical theory in a countable language is $\aleph_0$-stable, and so on (see [2]). A somewhat more involved preservation result was shown by the first author [3], namely that the theory of lovely pairs of models of a supersimple (respectively, superstable) theory is again supersimple (respectively, superstable).

With superstability and supersimplicity defined as above, the question whether a superstable theory with a generic automorphism is supersimple arises again. A specific instance of this question was asked by the second author and C. Ward Henson [10] regarding the theory of probability algebras equipped with a generic automorphism. It was answered negatively by the first author, showing that probability algebras with a generic automorphism are not superstable. The proof appears in Section 3.

However, the notions of $\aleph_0$-stability and/or superstability mentioned above might still be too strong: while they consider types of tuples up to arbitrarily small distance, one may further relax this and consider types also up to arbitrarily small perturbations of the entire language, or parts thereof. This idea can be formalised with the theory of perturbations as developed in [1] and somewhat restated in [4, Section 4]. We shall assume some familiarity with the second reference.

The goal of this paper is to study carefully two examples: Hilbert spaces and probability algebras, both equipped with a generic automorphism. The theory *APr* of atomless probability algebras and the theory *IHS* of infinite dimensional Hilbert spaces have some features in common. They are $\aleph_0$-stable, separably categorical over any finite set of parameters and types over sets are stationary. It follows (see [10]) that both IHS_τ and APr_τ admit model companions IHS_A and APr_A.

In Section 1 we deal with the theory IHS_A of Hilbert spaces equipped with a generic automorphism. We recall some if its properties from [8]. We use a Corollary of the Weyl-von Neumann-Berg Theorem to show that IHS_A is $\aleph_0$-stable up to perturbation (of the automorphism), and admits prime models up to perturbation over any set. Unlike the arguments in [8], our arguments can be extended to a generic action of a finitely generated group of automorphisms (i.e., a generic unitary representation, see [9]) and even to Hilbert spaces equipped with a generic action of a fixed finitely generated C^*-algebra. This section also serves as a soft analogue for the main results of the other sections.

In Section 2 we deal with the theory APr_A of probability algebras with a generic automorphism, first studied in [10]. Specifically, we show that APr_A is $\aleph_0$-stable up to perturbations of the automorphism. It is an open question if APr_A admits prime models up to perturbations.

In Section 3 we conclude with the first author's proof that without perturbation the theory APr_A is not superstable, showing that the results of Section 2 are in some sense optimal.

1 Hilbert spaces with an automorphism

Let us consider a Hilbert space $\mathcal{H}$ and let $B(\mathcal{H})$ denote the space of bounded linear operators on $\mathcal{H}$. We recall that the *operator norm* of $T \in B(\mathcal{H})$ is $\|T\| = \sup_{\|x\|=1} \|T(x)\|$. We also recall the notions of the *spectrum*, *punctual spectrum* and *essential spectrum* of an operator $T \in B(\mathcal{H})$:

$$\sigma(T) = \{\lambda \in \mathbb{C} \colon T - \lambda I \text{ is not invertible}\},$$

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \colon \ker(T - \lambda I) \neq 0\},$$

$$\sigma_e(T) = \{\text{non isolated points of } \sigma(T)\} \cup \{\lambda \in \mathbb{C} \colon \dim\ker(T - \lambda I) = \infty\}.$$

Definition 1.1 Let $\mathcal{H}$ be a Hilbert space, $T_0, T_1 \in B(\mathcal{H})$. We say that T_0 and T_1 are *approximately unitarily equivalent* if there is a sequence of unitary operators $\{U_n\}_{n\in\mathbb{N}}$ such that $\|T_0 - U_n T_1 U_n^*\| \to 0$.

Fact 1.2 (Weyl-von Neumann-Berg Theorem [12, p. 60]) *Let $\mathcal{H}$ be a Hilbert space and let $T_0, T_1 \in B(\mathcal{H})$ be normal operators. Then T_0 and T_1 are approximately unitarily equivalent if and only if*

(i) $\sigma_e(T_0) = \sigma_e(T_1)$

(ii) $\dim\ker(T_0 - \lambda I) = \dim\ker(T_1 - \lambda I)$ *for all* λ *in* $\mathbb{C} \setminus \sigma_e(T_0)$.

When considering a Hilbert space as a continuous structure we shall replace it with its unit ball, as described in [5]. We shall use the language $\mathcal{L} = \{0, -, \dot{2}, \frac{x+y}{2}\}$, where $\dot{2}x = \min(2, \frac{1}{\|x\|})x$ and $d(x,y) = \|\frac{x-y}{2}\|$. Notice that we can recover the norm as $\|x\| = d(x, -x)$. An axiomatisation for the class of (unit balls of) Banach spaces in this language, excluding the symbol $\dot{2}$, appears in [5]. The symbol $\dot{2}$ serves as a Skolem function for the fullness axiom there, yielding a universal theory. A Banach space is a Hilbert space if and only if the parallelogram identity holds, which is a universal condition as well:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

We obtain that the class of Hilbert spaces is elementary, admitting a universal theory *HS*. Its model companion is *IHS*, the theory of infinite dimensional Hilbert spaces, obtained by adding the appropriate scheme of existential conditions. It is easy to check that the theory *HS* has the amalgamation property, so *IHS* eliminates quantifiers (i.e., it is the model completion of *HS*).

Now let τ be a new unary function symbol and let $\mathcal{L}_\tau$ be $\mathcal{L} \cup \{\tau\}$. Let IHS_τ be the theory $IHS \cup \{\tau \text{ is an automorphism}\}$. Since *IHS* is $\aleph_0$-stable and separably categorical even after naming finitely many constants, the theory IHS_τ admits a model companion IHS_A (see [10]). The universal part of IHS_τ is $(IHS_\tau)^\forall = (HS_\tau)^\forall = HS \cup \{\tau \text{ is a linear and isometric}\}$. It is again relatively easy to check that $(HS_\tau)^\forall$ has the amalgamation property. Indeed, if $(\mathcal{H}_0, \tau_0) \subseteq (\mathcal{H}_i, \tau_i)$ for $i = 1, 2$ then we may write $(\mathcal{H}_i, \tau_i) = (\mathcal{H}_0, \tau_0) \oplus (\mathcal{H}'_i, \tau'_i)$, where $\oplus$ is the orthogonal direct sum, and then $(\mathcal{H}_0, \tau_0) \oplus (\mathcal{H}'_1, \tau'_1) \oplus (\mathcal{H}'_2, \tau'_2)$ will do. It thus follows that IHS_A eliminates quantifiers as well.

Proposition 1.3 (Ben Yaacov, Usvyatsov, Zadka [8]) *Let $\mathcal{H}$ be a separable Hilbert space and let τ be a unitary operator on $\mathcal{H}$. Then $(\mathcal{H}, \tau) \models IHS_A$ (i.e., $(\mathcal{H}, \tau)$ is existentially closed as a model of IHS_τ) if and only if $\sigma(\tau) = S^1$.*

Proof Clearly, if $(\mathcal{H}, \tau)$ is existentially closed, then $\sigma(\tau) = S^1$. On the other hand, assume that $(\mathcal{H}, \tau) \models IHS_\tau$ and that $\sigma(\tau) = S^1$. Passing to an elementary substructure, we may assume that $\mathcal{H}$ is separable. Now let $(\mathcal{H}_0, \tau_0)$ be separable and existentially closed. Since *IHS* is separably categorical, we may assume that $\mathcal{H}_0 = \mathcal{H}$. Since $\sigma(\tau_0) = \sigma(\tau) = S^1$, by Fact 1.2 there is a sequence $\{U_n\}_{n\in\omega}$ of unitary operators on

$\mathcal{H}$ such that $U_n\tau_1U_n^* \to \tau_0$ in norm. It follows that if $\mathcal{U}$ is a non-principal ultra-filter on $\mathbb{N}$ then $\Pi_{\mathcal{U}}(\mathcal{H}, U_n\tau U_n^*) = \Pi_{\mathcal{U}}(\mathcal{H}, \tau_0)$.

On the other hand, $(\mathcal{H}, U_n\tau U_n^*) \cong (\mathcal{H}, \tau)$ for all $n \in \mathbb{N}$. Thus $\Pi_{\mathcal{U}}(\mathcal{H}, \tau) \cong \Pi_{\mathcal{U}}(\mathcal{H}, \tau_0)$, whereby $(\mathcal{H}, \tau) \equiv (\mathcal{H}, \tau_0) \vDash IHS_A$. $\blacksquare_{1.3}$

Remark 1.4 Henson and Iovino observed that the theory IHS_A is not $\aleph_0$-stable (or even small) in the sense defined in the introduction. Indeed, let $(\mathcal{H}, \tau) \vDash IHS_A$ be $\aleph_1$-saturated and for each $\lambda \in S^1$ let $v_\lambda \in H$ be a normal vector such that $\tau v_\lambda = \lambda v_\lambda$. Then $d\big(\mathrm{tp}(v_\lambda), \mathrm{tp}(v_\rho)\big) = \sqrt{2}$ for $\lambda \neq \rho$. Thus the metric density character of $\mathrm{S}_1(\varnothing)$ is the continuum.

On the other hand, it is shown in [8] that IHS_A is superstable.

Let dcl_τ and acl_τ denote the definable and algebraic closure (in the real sort) in models of IHS_A. We claim that if $(\mathcal{H}, \tau) \vDash IHS_A$ and $A \subseteq H$, then $\mathrm{dcl}_\tau(A) = \mathrm{acl}_\tau(A) = \mathrm{dcl}\left(\bigcup_{n\in\mathbb{Z}} \tau^n(A)\right)$, where $\mathrm{dcl}(A)$ is the definable closure of A in the language $\mathcal{L}$. Indeed, let $B = \mathrm{dcl}\left(\bigcup_{n\in\mathbb{Z}} \tau^n(A)\right)$. Then clearly $B \subseteq \mathrm{dcl}_\tau(A)$. On the other hand, we may decompose $(\mathcal{H}, \tau) = (B, \tau{\restriction}_B) \oplus (B', \tau{\restriction}_{B'})$, in which case $(\mathcal{H}, \tau) \preceq (B, \tau{\restriction}_B) \oplus \bigoplus_{n\in\mathbb{N}} (B', \tau{\restriction}_{B'})$, showing that $\mathrm{acl}_\tau(A) \subseteq B$.

We may similarly characterise non forking in models of IHS_A. For $(\mathcal{H}, \tau) \vDash IHS_A$ and subsets $A, B, C \subseteq \mathcal{H}$, say that $A \mathop{\smile\!\!\!\!\!\!|}_B C$ if $P_{\mathrm{dcl}_\tau(B)}(a) = P_{\mathrm{dcl}_\tau(BC)}(a)$ for every $a \in A$. We leave it to the reader to check that $\mathop{\smile\!\!\!\!\!\!|}$ satisfies the usual axioms of a stable notion of independence (invariance, symmetry, transitivity, and so on), and therefore coincides with non forking.

Proposition 1.5 *Let $(\mathcal{H}, \tau) \vDash IHS_A$, $A, B \subseteq \mathcal{H}$. Then $\mathrm{tp}(A/B)$ is stationary and $\mathrm{Cb}(A/B)$ is inter-definable with the set $C = \{P_{\mathrm{dcl}_\tau(B)}(a)\})_{a\in A}$.*

Proof Stationarity follows from the characterisation of independence (and from quantifier elimination). It is also clear that $C \subseteq \mathrm{dcl}_\tau(B)$ and $A \mathop{\smile\!\!\!\!\!\!|}_C B$, and since we already know that $\mathrm{tp}(A/C)$ is stationary as well, we obtain $\mathrm{Cb}(A/B) \subseteq C$.

For the converse it suffices to show that for every $a \in A$, the projection $P_{\mathrm{dcl}_\tau(B)}(a)$ belongs to the definable closure of any Morley sequence in $\mathrm{tp}(A/B)$. So let $(A_n)_{n\in\mathbb{N}}$ be such a Morley sequence. Then $P_{\mathrm{dcl}_\tau(B)}(a_n) = P_{\mathrm{dcl}_\tau(B)}(a)$ for all $a \in A$ and all n, so $\{a_n - P_{\mathrm{dcl}_\tau(B)}(a)\}_{n\in\mathbb{N}}$ forms an orthogonal sequence of bounded norm. Thus

$$\sum_{n=0}^{m-1} \frac{a_n}{m} = P_{\mathrm{dcl}_\tau(B)}(a) + \sum_{n=0}^{m-1} \frac{a_n - P_{\mathrm{dcl}_\tau(B)}(a)}{m} \to P_{\mathrm{dcl}_\tau(B)}(a). \blacksquare_{1.5}$$

It follows that IHS_A has weak elimination of imaginaries, namely that for every imaginary element e there exists a real tuple (possibly infinite) A such that $A \subseteq \mathrm{acl}_\tau(e)$, $e \in \mathrm{dcl}^{eq}(A)$.

We now turn to perturbations of the automorphism in models of IHS_A. Let $(\mathcal{H}_i, \tau_i) \models IHS_A$ for $i = 0, 1$, and let $r \geq 0$. We define an *r-perturbation* of $(\mathcal{H}_0, \tau_0)$ to $(\mathcal{H}_1, \tau_1)$ to be an isometric isomorphism of Hilbert spaces $U\colon \mathcal{H}_0 \cong \mathcal{H}_1$ which satisfies in addition

$$\|U\tau_0 U^{-1} - \tau_1\| \leq r.$$

The set of all r-perturbations will be denoted $\mathrm{Pert}_r\big((\mathcal{H}_0, \tau_0), (\mathcal{H}_1, \tau_1)\big)$. It is fairly immediate to verify that this indeed satisfies all the conditions stated in [4, Theorem 4.4], and therefore does indeed correspond to a perturbation system as defined there.

Lemma 1.6 *Let $(\mathcal{H}_0, \tau_0) \subseteq (\mathcal{H}_i, \tau_i)$ be separable models of IHS_τ for $i = 1, 2$. Then we may write $(\mathcal{H}_i, \tau_i) = (\mathcal{H}_0, \tau_0) \oplus (\mathcal{H}'_i, \tau'_i)$, and let us assume that $\sigma(\tau'_1) \subseteq \sigma(\tau'_2)$ and that $\sigma(\tau'_1)$ has no isolated points.*

Then for every $\varepsilon > 0$ there is an isometric isomorphism $U\colon \mathcal{H}_1 \oplus \mathcal{H}'_2 \cong \mathcal{H}_2$, which fixes $\mathcal{H}_0$ such that $\|U(\tau_1 \oplus \tau'_2)U^{-1} - \tau_2\| \leq \varepsilon$.

Proof Under the assumptions we have $\sigma(\tau'_1 \oplus \tau'_2) = \sigma(\tau'_2)$. We also assume that $\sigma(\tau'_1)$ has no isolated points. Therefore, if $\lambda \in \sigma(\tau'_1 \oplus \tau'_2)$ is isolated then its eigenspace in $\mathcal{H}'_1 \oplus \mathcal{H}'_2$ is entirely contained in $\mathcal{H}'_2$, so the multiplicity (possibly infinite) of λ is the same for $\tau'_1 \oplus \tau'_2$ and for τ'_2. It follows that the hypotheses of Fact 1.2 hold, and we obtain $V\colon \mathcal{H}'_1 \oplus \mathcal{H}'_2 \cong \mathcal{H}'_2$ such that $\|V(\tau'_1 \oplus \tau'_2)V^{-1} - \tau'_2\| \leq \varepsilon$. Then $U = \mathrm{id}_{\mathcal{H}_0} \oplus V$ will do. $\blacksquare_{1.6}$

Theorem 1.7 *The theory IHS_A is $\aleph_0$-stable up to perturbation of the automorphism.*

Proof Let $(\mathcal{H}_0, \tau_0), (\mathcal{H}'_1, \tau'_1) \models IHS_A$ be separable, and let $(\mathcal{H}_1, \tau_1) = (\mathcal{H}_0, \tau_0) \oplus (\mathcal{H}'_1, \tau'_1)$. By Proposition 1.3 we have $(\mathcal{H}_1, \tau_1) \models IHS_A$, so $(\mathcal{H}_0, \tau_0) \preceq (\mathcal{H}_1, \tau_1)$ by model completeness. It will therefore be enough to show that every type over $\mathcal{H}_0$ is realised, up to perturbation, in $(\mathcal{H}_1, \tau_1)$. Such a type can always be realised in a separable elementary extension $(\mathcal{H}_2, \tau_2) \succeq (\mathcal{H}_1, \tau_1)$. Then $(\mathcal{H}_0, \tau_0) \subseteq (\mathcal{H}_2, \tau_2)$ and we may decompose the latter as $(\mathcal{H}_2, \tau_2) = (\mathcal{H}_0, \tau_0) \oplus (\mathcal{H}'_2, \tau'_2)$.

Notice that $(\mathcal{H}'_1, \tau'_1) \subseteq (\mathcal{H}'_2, \tau'_2)$, so $\sigma(\tau'_1) = \sigma(\tau'_2) = S^1$. We may therefore apply Lemma 1.6, obtaining for every $\varepsilon > 0$ there an isometric isomorphism $U_\varepsilon\colon \mathcal{H}_1 \to \mathcal{H}_2$ fixing $\mathcal{H}_0$ such that $\|\tau_1 - U_\varepsilon^{-1}\tau_2 U_\varepsilon\| < \varepsilon$.

We have thus shown that every type over $\mathcal{H}_0$ is realised, up to arbitrarily small perturbation of the automorphism, in a fixed separable extension $(\mathcal{H}_1, \tau_1) \succeq (\mathcal{H}_0, \tau_0)$, as desired. $\blacksquare_{1.7}$

Remark 1.8 Let G be a finitely generated discrete group and let IHS_{gG} be the theory of Hilbert spaces with a generic action of G by automorphism (see [9]). Using Voiculescu's Theorem [12] in place of Fact 1.2, the same argument shows that the theory IHS_{gG} is $\aleph_0$-stable up to perturbations of the automorphisms. This can even be further extended to the theory $IHS_{g\mathcal{A}}$ of a generic presentation of a finitely generated C^*-algebra $\mathcal{A}$.

Proposition 1.9 *The theory IHS_A has prime models up to perturbation over sets (of real or imaginary elements).*

By this we mean that for every set A in a model of IHS_A there exists a model $(\mathcal{H}_1, \tau_1)$, containing A, such that if $(\mathcal{H}_2, \tau_2)$ is any other model which contains A then, up to arbitrarily small perturbation of τ_2 to ρ_2, we can embed $(\mathcal{H}_1, \tau_1)$ elementarily in $(\mathcal{H}_2, \rho_2)$ over A.

Proof We may assume that the set A over which we seek a prime model is algebraically closed. By weak elimination of imaginaries we may assume that A is a real set, and we may further assume that $A = \mathrm{dcl}_\tau(A)$. It is therefore a Hilbert subspace $\mathcal{H}_0$ (possibly finite dimensional) on which $\tau_0 = \tau{\restriction}_{\mathcal{H}_0}$ is an automorphism. Moreover, since IHS_A eliminates quantifiers, the type of $\mathcal{H}_0$ is determined by the pair $(\mathcal{H}_0, \tau_0)$, and there is no need to consider the ambient structure.

If $(H_0, \tau_0) \vDash IHS_A$ there is nothing to prove. Otherwise $\sigma(\tau_0) \subsetneq S^1$. Let $(H_1', \tau_1') \vDash IHS_\tau$ be separable such that $\sigma(\tau_1') = \overline{S^1 \smallsetminus \sigma(\tau_0)}$ (for example we may take $H_1' = L_2(S^1 \smallsetminus \sigma(\tau_0))$ in the Lebesgue measure, with $(\tau_1' f)(x) = xf(x)$). Let $(\mathcal{H}_1, \tau_1) = (\mathcal{H}_0, \tau_0) \oplus (\mathcal{H}_1', \tau_1')$. Clearly $\sigma(\tau_1) = S^1$, so $(\mathcal{H}_1, \tau_1) \vDash IHS_A$, and we shall prove that it is prime, up to perturbation of the automorphism, over $\mathcal{H}_0$.

So let $(\mathcal{H}_0, \tau_0) \subseteq (\mathcal{H}_2, \tau_2) \vDash IHS_A$ and we may assume that $\mathcal{H}_2$ is separable. As usual, we may decompose it as $(\mathcal{H}_2, \tau_2) = (\mathcal{H}_0, \tau_0) \oplus (\mathcal{H}_2', \tau_2')$. Since $\sigma(\tau_2) = S^1$, we necessarily have $\sigma(\tau_2') \supseteq S^1 \smallsetminus \sigma(\tau_0)$, and since $\sigma(\tau_2')$ is moreover closed, it contains $\sigma(\tau_1)$. By Lemma 1.6, for every $\varepsilon > 0$ there exists an isometric isomorphism $U\colon \mathcal{H}_1 \oplus \mathcal{H}_2' \cong \mathcal{H}_2$ fixing $\mathcal{H}_0$ such that $\|U(\tau_1 \oplus \tau_2')U^{-1} - \tau_2\| \leq \varepsilon$. By Proposition 1.3 we also have $(\mathcal{H}_1, \tau_1) \preceq (\mathcal{H}_1, \tau_1) \oplus (\mathcal{H}_2', \tau_2')$

Thus $\rho_2 = U(\tau_1 \oplus \tau_2')U^{-1}$ is as desired. $\blacksquare_{1.9}$

2 Probability algebras with an automorphism

By a probability space we mean a triplet $(X, \mathcal{B}, \mu)$, where X is a set, $\mathcal{B}$ a σ-algebra of subsets of X, μ a σ-additive positive measure on $\mathcal{B}$ such that $\mu(X) = 1$. A probability space $(X, \mathcal{B}, \mu)$ is called *atomless* if for every $A \in \mathcal{B}$ there is $C \in \mathcal{B}$ such that $\mu(A \cap C) = \frac{1}{2}\mu(A)$. We say that two elements $A, B \in \mathcal{B}$ determine the same *event*, and write $A \sim_\mu B$ if $\mu(A \triangle B) = 0$. The relation $\sim_\mu$ is an equivalence relation and the collection of classes is denoted by $\overline{B}$ and is called the *measure algebra* associated to $(X, \mathcal{B}, \mu)$. Operations such as unions, interesections and complements are well defined for events, as well as the measure. The distance between two events $a, b \in \overline{B}$ is given by the measure of their symmetric difference. This renders $\overline{\mathcal{B}}$ a complete metric space.

Conversely, let $(\mathscr{B}, 0, 1, \cdot^c, \cup, \cap)$ be a Boolean algebra and assume that d is a complete metric on $\mathscr{B}$. Let $\mu(x)$ be an abbreviation for $d(0, x)$ and assume furthermore that $d(x, y) = \mu(x \triangle y)$, $\mu(x) + \mu(y) = \mu(x \cap y) + \mu(x \cup y)$ and $\mu(1) = 1$. Then $\mathscr{B}$ is the probability algebra associated to some probability space (and we may moreover take that space to be the Stone space of $\mathscr{B}$, equipped with the Borel σ-algebra).

We may view probability algebras as continuous structures in the language $\mathcal{L}_{Pr} = \{0, 1, \cdot^c, \cup, \cap\}$ (the distance symbol is always implicit, and the measure can be recovered from it as above). The class of probability algebras is elementary and admits a universal theory denoted Pr. Its model completion is APr, the theory of atomless probability algebras. It admits quantifier elimination, is $\aleph_0$-categorical (even over finitely many parameters) and $\aleph_0$-stable (see [7, 10]).

Definition 2.1 Let $\mathscr{B}$ be a probability algebra. An automorphism $\tau \in \mathrm{Aut}(\mathscr{B})$ is said to be *aperiodic* if for every non-zero event $a \in \mathscr{B}$ and every $n > 0$ there is a sub-event $b \subseteq a$ such that $\tau^n(b) \neq b$. (In other words, the *support* of τ^n is 1 for all $n \geq 1$.)

Fact 2.2 (Halmos-Rokhlin-Kakutani Lemma, [13, 386C]) *Let $\mathscr{B}$ be a probability algebra, $\tau \in \mathrm{Aut}(\mathscr{B})$. Then τ is aperiodic if and only if, for every $n \geq 1$ and every $\varepsilon > 0$ there is $a \in \mathscr{B}$ such that $a, \tau a, \ldots, \tau^{n-1}(a)$ are disjoint and $n\mu(a) > 1 - \varepsilon$.*

Now let $\mathcal{L}_\tau = \mathcal{L}_{Pr} \cup \{\tau\}$ where τ is a new unary function symbol. Let APr_τ be the theory $APr \cup \{\tau \text{ is an automorphism}\}$. It was shown in [10] that APr_τ admits a model companion APr_A, consisting of APr_τ together with axioms saying that τ is aperiodic.

Definition 2.3 By the *Lebesgue space* we mean the probability space $([0, 1], \lambda)$, where λ is the standard Lebesgue measure. The associated probability algebra $\mathscr{L} = \mathfrak{B}([0, 1], \lambda)$ is the unique separable atomless probability algebra.

An *automorphism* τ of the Lebesgue space is a measurable, measure-preserving bijection between measure one subsets of $[0,1]$.

Remark 2.4 (i) Every automorphism of the probability algebra $\mathscr{L}$ comes from an automorphism of the Lebesgue space.

(ii) An automorphism τ of the Lebesgue space induces an aperiodic automorphism on $\mathscr{L}$ if and only if τ itself is aperiodic, namely if $\lambda\{x \in [0,1] \colon \tau^n(x) = x\} = 0$.

Definition 2.5 Let $\mathscr{A}$ be a probability algebra. We equip $\mathrm{Aut}(\mathscr{A})$ with the uniform convergence metric

$$d(\tau_0, \tau_1) = \sup_{x \in \mathscr{A}} d(\tau_0(x), \tau_1(x)).$$

Let $(\mathscr{A}_i, \tau_i) \vDash APr_A$ for $i = 0, 1$ and let $r \geq 0$. Then an *r-perturbation* of $(\mathscr{A}_0, \tau_0)$ to $(\mathscr{A}_1, \tau_1)$ is an (isometric) isomorphism $f \colon \mathscr{A}_0 \cong \mathscr{A}_1$ such that $d\big(f\tau_0 f^{-1}, \tau_1\big) \leq r$.

Notice that this is essentially the same definition as for (unit balls of) Hilbert space. In particular, as in the Hilbert space case, this definition satisfies the conditions of [4, Theorem 4.4] and thereby comes from a perturbation system.

Definition 2.6 Let $\mathscr{A}$ be a probability algebra, $\tau \in \mathrm{Aut}(\mathscr{A})$, $a \in \mathscr{A}$. We say that (a, τ) *generate an (n, ε)-partition (of $\mathscr{A}$)* if $a, \tau(a), \ldots, \tau^{n-1}(a)$ are disjoint and $\mu\left(\bigvee_{i<n} \tau^i(a)\right) \geq 1 - \varepsilon$. An $(n, 0)$-partition will simply be called an *n-partition*.

If $\mathscr{A}_i$, $i = 0, 1$ are probability algebras and $\mathscr{B} = \mathscr{A}_0 \otimes \mathscr{A}_1$ is their free amalgam, we may identify $\mathscr{A}_0$ with its image $\mathscr{A}_0 \otimes 1 = \{a \otimes 1 \colon a \in \mathscr{A}_0\} \subseteq \mathscr{B}$, and similarly $\mathscr{A}_1 \cong 1 \otimes \mathscr{A}_1 \subseteq \mathscr{B}$. In particular, if $(\mathscr{A}_i, \tau_i) \vDash APr_\tau$, $i = 0, 1$ then $(\mathscr{A}_0 \otimes \mathscr{A}_1, \tau_0 \otimes \tau_1) \vDash APr_\tau$ as well. If in addition $(\mathscr{A}_0, \tau_0) \vDash APr_A$ then τ_0 is aperiodic, whereby so is $\tau_0 \otimes \tau_1$, i.e., $(\mathscr{A}_0 \otimes \mathscr{A}_1, \tau_0 \otimes \tau_1) \vDash APr_A$. Since APr_A is model complete we conclude that

$$(\mathscr{A}_0, \tau_0) \vDash APr_A,\ (\mathscr{A}_1, \tau_1) \vDash APr_\tau \quad \Longrightarrow \quad (\mathscr{A}_0, \tau_0) \preceq (\mathscr{A}_0 \otimes \mathscr{A}_1, \tau_0 \otimes \tau_1).$$

Definition 2.7 Let $(\mathscr{A}, \tau_{\mathscr{A}}) \preceq (\mathscr{B}, \tau_{\mathscr{B}}) \vDash APr_A$ be separable. We say that $(\mathscr{B}, \tau_{\mathscr{B}})$ is *partitioned over $\mathscr{A}$* if:

(i) The probability algebra $\mathscr{B}$ is isomorphic to $\mathscr{A} \otimes \mathscr{L}$ over $\mathscr{A}$ (meaning that $a \in \mathscr{A}$ gets mapped to $a \otimes 1$).

(ii) Under this isomorphism, for each $0 < n \in \mathbb{N}$ there exists $c_n \in \mathscr{L}$ such that $(1 \otimes c_n, \tau_{\mathscr{B}})$ generate an n-partition, all of whose members belong to $1 \otimes \mathscr{L}$.

Lemma 2.8 *Let* $(\mathscr{A}, \tau_{\mathscr{A}}) \preceq (\mathscr{B}, \tau_{\mathscr{B}}) \models APr_A$ *be separable. Then there exists a further elementary extension* $(\mathscr{B}, \tau_{\mathscr{B}}) \preceq (\mathscr{C}, \tau_{\mathscr{C}})$ *which is partitioned over* $\mathscr{A}$.

Proof First of all, we may assume that $\mathscr{B}$ is atomless over $\mathscr{A}$. Indeed, $(\mathscr{B}', \tau'_{\mathscr{B}}) = (\mathscr{B}, \tau_{\mathscr{B}}) \otimes (\mathscr{L}, \mathrm{id})$ is a separable elementary extension of $(\mathscr{B}, \tau_{\mathscr{B}})$ and we may replace the latter with the former. Therefore we may assume that $\mathscr{B} = \mathscr{A} \otimes \mathcal{L}$.

It is not difficult to construct an automorphism $\rho \in \mathrm{Aut}(\mathscr{L})$ such that for each n there is $c_n \in \mathscr{L}$ such that (c_n, ρ) generate an n-partition. Let $(\mathscr{C}, \tau_{\mathscr{C}}) = (\mathscr{B} \otimes \mathscr{L}, \tau_{\mathscr{B}} \otimes \rho)$, so $(\mathscr{B}, \tau_{\mathscr{B}}) \preceq (\mathscr{C}, \tau_{\mathscr{C}})$.

On the other hand we have $\mathscr{C} = \mathscr{B} \otimes \mathscr{L} = \mathscr{A} \otimes [\mathscr{L} \otimes \mathscr{L}]$, and for every $k < n$

$$\tau_{\mathscr{C}}^k(1 \otimes [1 \otimes c_n]) = 1 \otimes [1 \otimes \rho^k(c_n)].$$

Now use the fact that $\mathscr{L} \otimes \mathscr{L} \cong \mathscr{L}$ to conclude. $\blacksquare_{2.8}$

Notation 2.9 For a probability algebra $\mathscr{C}$ and $c \in \mathscr{C}$, let $\mathscr{C}_{\leq c}$ denote the ideal $\{c' \in \mathscr{C} : c' \leq c\}$.

Notation 2.10 For $0 < n \in \mathbb{N}$ fix $(\ell_n, \rho_n) \in \mathscr{L} \times \mathrm{Aut}(\mathscr{L})$ generating an n-partition such that in addition $(\rho_n)^n = \mathrm{id}$. Note that this determines $(\mathscr{L}, \rho_n)$ up to isomorphism, and that it is *not* a model of APr_A.

Lemma 2.11 *Let* $(\mathscr{A}, \tau_{\mathscr{A}}) \preceq (\mathscr{B}, \tau_{\mathscr{B}}) \models APr_A$*, and assume that* $(\mathscr{B}, \tau_{\mathscr{B}})$ *is partitioned over* $\mathscr{A}$*. Then for every* $0 < n \in \mathbb{N}$ *there exists* $\tau'_{\mathscr{B}} \in \mathrm{Aut}(\mathscr{B})$ *such that:*

(i) $d(\tau_{\mathscr{B}}, \tau'_{\mathscr{B}}) \leq \frac{1}{2n}$.

(ii) $(\mathscr{B}, \tau'_{\mathscr{B}}) \cong (\mathscr{A}, \tau_{\mathscr{A}}) \otimes (\mathscr{L}, \rho_n)$ *over* $\mathscr{A}$*, where* $(\mathcal{L}, \rho_n)$ *are as in Notation 2.10.*

Proof We may assume that $(\mathscr{B}, \tau_{\mathscr{B}}) = (\mathscr{A} \otimes \mathscr{L}, \tau_{\mathscr{B}})$ and that this identification witnesses that $(\mathscr{B}, \tau_{\mathscr{B}})$ is partitioned over $\mathscr{A}$. Therefore there exists $c \in \mathscr{L}$ are such that $(1 \otimes c, \tau_{\mathscr{B}})$ generate an n-partition all of whose members are in $1 \otimes \mathscr{L}$. Let $1 \otimes c_k = \tau_{\mathscr{B}}^k(1 \otimes c) \in 1 \otimes \mathcal{L}$ for $k < n$ (or, for that matter, for all $k \in \mathbb{Z}$).

Let ℓ_n, ρ_n be as in Notation 2.10. Since $c, \ell_n \in \mathscr{L}$ and $\mu(c) = \mu(\ell_n) = \frac{1}{n}$, there is an isomorphism $\theta_0 \colon \mathscr{L}_{\leq c} \cong \mathscr{L}_{\leq \ell_n}$, which induces in turn $\theta_1 = \mathrm{id}_{\mathscr{A}} \otimes \theta_0 \colon \mathscr{B}_{\leq 1 \otimes c} \cong \mathscr{B}_{\leq 1 \otimes \ell_n}$. We shall extend θ_1 to an automorphism of $\mathscr{B}$ as follows. For $b \in \mathscr{B}$, observe that $b = \bigvee_{k<n} \big(b \wedge (1 \otimes c_k)\big)$ is a partition of b, and $\tau_{\mathscr{B}}^{-k}\big(b \wedge (1 \otimes c_k)\big) \in \mathscr{B}_{\leq 1 \otimes c}$. We then define

$$\theta_2(b) = \bigvee_{k<n} (\tau_{\mathscr{A}} \otimes \rho_n)^k \theta_1 \tau_{\mathscr{B}}^{-k}\big(b \wedge (1 \otimes c_k)\big).$$

For each $k < n$, θ_2 restricts to an isomorphism $\mathscr{B}_{\leq 1\otimes c_k} \cong \mathscr{B}_{\leq 1\otimes \rho_n^k(\ell_n)}$, so $\theta_2 \in \mathrm{Aut}(\mathscr{B})$. In addition, if $a \in \mathscr{A}$ then:

$$\begin{aligned}\theta_2\big((a\otimes 1)\wedge(1\otimes c_k)\big) &= (\tau_{\mathscr{A}}\otimes\rho_n)^k\theta_1\tau_{\mathscr{B}}^{-k}(a\otimes c_k)\\ &= (\tau_{\mathscr{A}}\otimes\rho_n)^k\theta_1\big(\tau_{\mathscr{A}}^{-k}(a)\otimes c\big)\\ &= (\tau_{\mathscr{A}}\otimes\rho_n)^k\big(\tau_{\mathscr{A}}^{-k}(a)\otimes \ell_n\big)\\ &= a\otimes\rho_n^k(\ell_n),\end{aligned}$$

whereby $\theta_2(a\otimes 1) = a\otimes 1$. Thus θ_2 acts as the identity on $\mathscr{A}\otimes 1$.

Let $\tau'_{\mathscr{B}} = \theta_2^{-1}(\tau_{\mathscr{A}}\otimes\rho_n)\theta_2$. For $b\in\mathscr{B}$ and for $k<n$ let $b_k = b\wedge(1\otimes c_k)\in\mathscr{B}_{\leq 1\otimes c_k}$, in which case $\theta_2(b_k)\in\mathscr{B}_{\leq 1\otimes\rho_n^k(\ell_n)}$ and $(\tau_{\mathscr{A}}\otimes\rho_n)\theta_2(b_k)\in\mathscr{B}_{\leq 1\otimes\rho_n^{k+1}(\ell_n)}$. Assume now in addition that $0\leq k\leq n-2$, i.e., that $k+1\leq n-1$. Then in the expression $\tau'_{\mathscr{B}}(b_k) = \theta_2^{-1}(\tau_{\mathscr{A}}\otimes\rho_n)\theta_2(b_k)$, the instances of θ_2^{-1} and θ_2 can be expanded explicitly as follows:

$$\begin{aligned}\tau'_{\mathscr{B}}(b_k) &= \theta_2^{-1}(\tau_{\mathscr{A}}\otimes\rho_n)\theta_2(b_k)\\ &= \big[\tau_{\mathscr{B}}^{k+1}\theta_1^{-1}(\tau_{\mathscr{A}}\otimes\rho_n)^{-k-1}\big](\tau_{\mathscr{A}}\otimes\rho_n)\big[(\tau_{\mathscr{A}}\otimes\rho_n)^k\theta_1\tau_{\mathscr{B}}^{-k}\big](b_k)\\ &= \tau_{\mathscr{B}}(b_k).\end{aligned}$$

(On the other hand, this fails for $k = n-1$, since then θ_2^{-1} expands to θ_1^{-1}.) It follows that $d\big(\tau_{\mathscr{B}}(b),\tau'_{\mathscr{B}}(b)\big) = d\big(\tau_{\mathscr{B}}(b_{n-1}),\tau'_{\mathscr{B}}(b_{n-1})\big)\leq\frac{1}{2n}$ (since both events are sub-events of same measure of $1\otimes c_{n-1}$ which has measure $\frac{1}{n}$). Thus $d(\tau_{\mathscr{B}},\tau'_{\mathscr{B}})\leq\frac{1}{2n}$, as desired. $\blacksquare_{2.11}$

Theorem 2.12 *The theory APr_A is $\aleph_0$-stable up to perturbations of the automorphism.*

Proof Let $(\mathscr{A},\tau_{\mathscr{A}})\vDash APr_A$. By Lemma 2.8 $(\mathscr{A},\tau_{\mathscr{A}})$ admits a partitioned extension $(\mathscr{B},\tau_{\mathscr{B}})$. We claim that up to perturbation, every type over $\mathscr{A}$ is realised in $(\mathscr{B},\tau_{\mathscr{B}})$.

Indeed any such type is realised in some separable extension $(\mathscr{C},\tau_{\mathscr{C}})\succeq(\mathscr{A},\tau_{\mathscr{A}})$. By Lemma 2.8 again we may assume that $(\mathscr{C},\tau_{\mathscr{C}})$ is also partitioned over $\mathscr{A}$. By Lemma 2.11 any two partitioned extensions of $\mathscr{A}$ admit an $\frac{1}{2n}$ perturbation over $\mathscr{A}$ to $(\mathscr{A},\tau_{\mathscr{A}})\otimes(\mathscr{L},\rho_n)$, and composing these we obtain an $\frac{1}{n}$-perturbation $(\mathscr{C},\tau_{\mathscr{C}})\to(\mathscr{B},\tau_{\mathscr{B}})$ which fixes $\mathscr{A}$, and this for every $0<n\in\mathbb{N}$. $\blacksquare_{2.12}$

3 Non superstability of probability algebras with a generic automorphism

The second author and Henson [10] asked whether the theory of probability algebras with a generic automorphism is superstable, as one might expect by analogy with a theorem of Chatzidakis and Pillay [11] regarding generic automorphisms in classical first order logic. In this section we present the first author's negative answer, which chronologically came before the results in Section 2, and to a large extent motivated them.

Our aim is to show that the theory APr_A admits many types over small sets of parameters, and for this purpose it will suffice to show that there are many 1-types over parameters which belong to the fixed algebra of the automorphism. We therefore proceed in two steps, first characterising such types and then showing there are many of them. Throughout we let $(\mathcal{U}, \tau)$ denote a monster model of APr_A, and let $\mathcal{U}^\tau$ denote the fixed sub-algebra of $\mathcal{U}$ under τ.

3.1 Types over the empty set and over fixed sub-algebras

Let us try to describe the space of 1-types in APr_A over a set of parameters contained in $\mathcal{U}^\tau$. We start with types over the empty set.

Consider a function $\eta\colon 2^{<\omega} \to [0,1]$ sending $s \mapsto \eta_s$. We call such a function *shift invariant* if

$$\text{(SI)} \qquad \eta_\varnothing = 1, \qquad \eta_{s\frown 0} + \eta_{s\frown 1} = \eta_{0\frown s} + \eta_{1\frown s} = \eta_s, \qquad \text{for all } s \in 2^{<\omega}.$$

We define $X \subseteq [0,1]^{2^{<\omega}}$ to consist of all shift invariant mappings. This is a closed subset of $[0,1]^{2^{<\omega}}$, and therefore compact.

Let $p \in \mathrm{S}_1(APr_A)$, and for $n \in \mathbb{N}$, $s \in 2^n$, let $\eta_{p,s} = \mu\left(\bigwedge_{i<n} \tau^i(x^{s_i})\right)^p$ (where $x^0 = x$, $x^1 = x^c$ is the natural action of $(\mathbb{Z}/2\mathbb{Z}, +)$). Then $\eta_p\colon s \mapsto \eta_{p,s}$ is shift invariant, yielding a mapping $\rho\colon S_1(APr_A) \to X$ sending $p \mapsto \eta_p$. This mapping ρ is clearly continuous, and by quantifier elimination it is injective.

Conversely, let $\eta \in X$, and let $\mathscr{A}$ be any sufficiently homogeneous atomless probability algebra. Then one can find in $\mathscr{A}$ a sequence of events (a_n) such that $\mu(a_0) = \eta_0$, $\mu(a_0)^c = \eta_1 = 1 - \eta_0$, and in general, for every n and $s \in 2^n$:

$$\mu\Big(a_n \wedge \bigwedge_{k<n} a_k^{s_k}\Big) = \eta_{s\frown 0}, \qquad \mu\Big(a_n^c \wedge \bigwedge_{k<n} a_k^{s_k}\Big) = \eta_{s\frown 1} = \eta_s - \eta_{s\frown 0}.$$

This is indeed consistent by shift invariance. Moreover, shift invariance implies that for every $n, k \in \mathbb{N}$ and $s \in 2^n$: $\eta_s = \mu\left(\bigwedge_{i<n} a_{k+i}^{s_i}\right)$ (by induction on k). It follows by quantifier elimination that the mapping $a_n \mapsto a_{n+1}$ is elementary and therefore extends to an automorphism $\tau_{\mathscr{A}} \in \mathrm{Aut}(\mathscr{A})$, and we may embed $(\mathscr{A}, \tau_{\mathscr{A}})$ in $(\mathcal{U}, \tau)$. In other words, for every $\eta \in X$ we can find $a \in \mathcal{U}$ such that $\eta_s = \mu\left(\bigwedge_{i<n} \tau^i(a^{s_i})\right)$ for all

$s \in 2^{<\omega}$. Thus $\eta = \rho(\mathrm{tp}(a))$, showing that ρ is bijective. Since it is also continuous, from a compact to a Hausdorff space, it is a homeomorphism.

If Y is an arbitrary topological space we have $C(Y,[0,1])^{2^{<\omega}} = C\big(Y,[0,1]^{2^{<\omega}}\big)$ as sets. Equipping $C(Y,Z)$ with the compact-open topology and $2^{<\omega}$ with the discrete topology these are homeomorphisms. (The common topology can be given by a sub-basis, where a sub-basic open set is of the form $\{f \in C(Y \times 2^{<\omega},[0,1]) \colon f[K \times \{s\}] \subseteq U\}$, with $K \subseteq Y$ compact, $s \in 2^{<\omega}$ and $U \subseteq [0,1]$ is open). We may define when a mapping $\eta \in C(Y,[0,1])^{2^{<\omega}}$ is shift invariant by (SI) as above, and let X_Y be the set of all such shift invariant functions. It is then clear that $X_Y = C(Y,X)$.

We now turn to types over a sub-algebra $\mathscr{A} \subseteq \mathcal{U}^\tau$, namely over parameters fixed by τ. We shall use the following.

Fact 3.1 *Let $\mathscr{A}$ be a probability algebra, and let $\widetilde{\mathscr{A}}$ be the Stone space of the underlying Boolean algebra. For an event $a \in \mathscr{A}$ let $\tilde{a} \subseteq \widetilde{\mathscr{A}}$ be the corresponding clopen set.*

(i) *The space $\widetilde{\mathscr{A}}$ admits a unique regular Borel probability measure $\tilde{\mu}$ such that $\tilde{\mu}(\tilde{a}) = \mu(a)$ for all $a \in \mathscr{A}$.*

(ii) *The probability algebra of $(\widetilde{\mathscr{A}},\tilde{\mu})$ is canonically isomorphic to $\mathscr{A}$, identifying the equivalence class of $\tilde{a}$ with the event a.*

(iii) *The natural mapping $C(\widetilde{\mathscr{A}},\mathbb{R}) \to L_\infty(\widetilde{\mathscr{A}},\tilde{\mu})$ is bijective. In other words, every equivalence class of bounded Borel functions up to equality $\tilde{\mu}$-a.e. contains a unique continuous representative.*

(iv) *Let $\mathscr{B} \supseteq \mathscr{A}$ be a larger probability algebra. Then there exists a conditional expectation operation $\mathbb{E}[\cdot|\mathscr{A}] \colon L_1(\widetilde{\mathscr{B}}) \to L_1(\widetilde{\mathscr{A}})$ where $\mathbb{E}[f|\mathscr{A}]$ is the unique function such that for all $a \in \mathscr{A}$:*

$$\int_{\tilde{a}\subseteq\widetilde{\mathscr{B}}} f\, d\tilde{\mu}_{\mathscr{B}} = \int_{\tilde{a}\subseteq\widetilde{\mathscr{A}}} \mathbb{E}[f|\mathscr{A}]\, d\tilde{\mu}_{\mathscr{A}}.$$

Proof See [13]. The Stone space associated to a Boolean algebra is discussed in 311. The construction of the measure $\tilde{\mu}$ and the isomorphism between $\mathscr{A}$ and the probability algebra of $(\widetilde{\mathscr{A}},\tilde{\mu})$ appears in 321J. The identification of $C(\widetilde{\mathscr{A}},\mathbb{R})$ and $L_\infty(\widetilde{\mathscr{A}},\tilde{\mu})$ can be found in 363 and 364K. Finally, conditional expectations are discussed in 365R. $\blacksquare_{3.1}$

Now let $\mathscr{A} \subseteq \mathcal{U}^\tau \subseteq \mathcal{U}$, and let $p \in \mathrm{S}_1(\mathscr{A})$, say realised by $b \in \mathcal{U}$. For $s \in 2^n$, the type p determines the mapping associating to every $a \in \mathscr{A}$ the measure $\mu\left(a \wedge \bigwedge_{i<n} \tau^i(b^{s_i})\right)$. In other words, p determines the function $\eta_{p,s} =$

$\mathbb{P}\Big[\bigwedge_{i<n}\tau^i(b^{s_i})\Big|\mathscr{A}\Big] \in L_1(\widetilde{\mathscr{A}})$. Since the essential range of $\eta_{p,s}$ lies in $[0,1]$ we have $\eta_{p,s} \in L_\infty(\widetilde{\mathscr{A}}) = C(\widetilde{\mathscr{A}},\mathbb{R})$, and in fact $\eta_{p,s} \in C(\widetilde{\mathscr{A}},[0,1])$. Let $\eta_p \in C(\widetilde{\mathscr{A}},[0,1])^{2^{<\omega}}$, $\eta_p\colon s \mapsto \eta_{p,s}$. It is not difficult to see that η_p satisfies (SI), so identifying $C(\widetilde{\mathscr{A}},[0,1])^{2^{<\omega}}$ with $C(\widetilde{\mathscr{A}},[0,1]^{2^{<\omega}})$ we actually have $\eta_p \in X_{\mathscr{A}} := C(\tilde{\mathscr{A}},X)$. We have thus obtained a mapping $\rho_{\mathscr{A}}\colon \mathrm{S}_1(\mathscr{A}) \to X_{\mathscr{A}}$. Again, it is injective by quantifier elimination and continuous, and a construction as above yields that it is surjective. We have thus obtained a homeomorphism

$$\rho_{\mathscr{A}}\colon \mathrm{S}_1(\mathscr{A}) \cong X_{\mathscr{A}} = C(\widetilde{\mathscr{A}},X).$$

For a closed set $K \subseteq X$ and $a \in \mathscr{A}$ define

$$K^a = \{\eta \in X_{\mathscr{A}}\colon \eta[\tilde{a}] \subseteq K\}.$$

It is not difficult to see that K^a is closed in the compact-open topology. If $\pi(x)$ is a partial type over $\varnothing$ and $K \subseteq X$ corresponds to the closed set $[\pi] \subseteq \mathrm{S}_1(APr_A)$, then the closed set K^a corresponds to $[\pi^a] \subseteq \mathrm{S}_1(\mathscr{A})$, where π^a is a partial type over $\mathscr{A}$. If $\pi^a(b)$ holds we say that *b satisfies π over a*.

Let $d_{\mathscr{A}}$ denote the distance function between types over $\mathscr{A}$, and similarly $d_\varnothing$. It is fairly easy to verify that the distance mapping $d_\varnothing\colon \mathrm{S}_1(APr_A)^2 \to [0,1]$, i.e., $d_\varnothing\colon X^2 \to [0,1]$, is Borel measurable (but not continuous, since the theory is not $\aleph_0$-categorical). Thus, if $p,q \in \mathrm{S}_1(\mathscr{A})$, then $d_\varnothing \circ (\eta_p,\eta_q)$ is a random variable from $\mathscr{A}$ to $[0,1]$, which we can integrate.

Lemma 3.2 *For all $p,q \in \mathrm{S}_1(\mathscr{A})$: $d_{\mathscr{A}}(p,q) \geq \int d_\varnothing \circ (\eta_p,\eta_q)\,d\tilde{\mu}$.*

Proof Assume $a \vDash p$, $b \vDash q$. Let $g = \mathbb{P}[a \triangle b|\mathscr{A}]$. Then $g \geq d_\varnothing \circ (\eta_p,\eta_q)$. $\blacksquare_{3.2}$

3.2 Non superstability proof

We say that a continuous theory is *small* if the metric on $\mathrm{S}_n(APr)$ is separable for all $n \in \mathbb{N}$.

Lemma 3.3 *The theory APr_A is not small. More precisely, there is an uncountable family of types over the empty set every two of which have distance $\geq \frac{1}{2}$.*

Proof For every real α, let p_α be the type of one half of the circle on which τ acts by rotation by $2\pi\alpha$ (the measure on the circle being the Lebesgue measure normalised to have total length 1). If $\alpha, \beta \geq 0$ are irrational and linearly independent over the rationals then for every $\varepsilon > 0$ there exist $n, k, \ell \in \mathbb{N}$ such that $|n\alpha - k|, |n\beta - \ell - \frac{1}{2}| < \varepsilon$. If $a \vDash p_\alpha$, $b \vDash p_\beta$ then $d(a, \tau^n(a)) = \mu(a \triangle \tau^n(a)) < 2\varepsilon$ while $d(b, \tau^n(b)) > 1 - 2\varepsilon$. It follows that $2d(a,b) = d(a,b) + d(\tau^n(a), \tau^n(b)) \geq 1 - 4\varepsilon$, namely $d(a,b) \geq \frac{1}{2} - \varepsilon$. Therefore $d(p_\alpha, p_\beta) \geq \frac{1}{2}$ (it is not difficult to check that the distance between the types is in fact equal to $\frac{1}{2}$).

Let $S \subseteq \mathbb{R}$ be a vector space base for $\mathbb{R}$ over $\mathbb{Q}$. Then follows that $\{p_\alpha : \alpha \in S\}$ is a continuum-size set of equally distanced types. $\blacksquare_{3.3}$

Proposition 3.4 *There exists a family $\{\pi_s(x)\}_{s \in 2^{<\omega}}$ of partial types over $\varnothing$ such that*

(i) *For all $s \in 2^{<\omega}$ we have $d(\pi_{s\frown 0}, \pi_{s \frown 1}) \geq \frac{1}{3}$, meaning that $d(a_0, a_i) \geq \frac{1}{3}$ whenever $a_i \vDash \pi_{s\frown i}$.*

(ii) *If $s, t \in 2^{<\omega}$ and t extends s then $\pi_t \vdash \pi_s$.*

Proof This is a metric Cantor-Bendixson rank argument which applies more generally, saying that if T is a non-small theory with a countable language then such a tree exists (with $\frac{1}{3}$ possibly replaced with another positive constant). For an even more general statement of this fact see [4, Propositions 3.16 and 3.19].

Define compacts subsets $X_\alpha \subseteq \mathrm{S}_1(\varnothing)$ by induction on α. Start with $X_0 = \mathrm{S}_1(\varnothing)$; for α limit, $X_\alpha = \bigcap_{\beta < \alpha} X_\beta$; and given X_α, obtain $X_{\alpha+1}$ by removing from X_α all points for which there is a relatively open neighbourhood of diameter $< \frac{1}{2}$.

Since the language is countable, the topology on X admits a countable base. If we only take out basic open sets of diameter $< \frac{1}{2}$ we still get the same sequence X_α, and since the base is countable the sequence stabilises before $\aleph_1$. Let $S \subseteq \mathrm{S}_1(\varnothing)$ be an uncountable subset of types every pair of which have distance $\geq \frac{1}{2}$. Each set of diameter $< \frac{1}{2}$ can contain at most one member of S, so $X_{\aleph_1} \neq \varnothing$. The topological space $X_{\aleph_1}$ (with the induced topology) is $\frac{1}{2}$-perfect, meaning that every non-empty open subset has diameter $\geq \frac{1}{2}$.

Let $D = \{(q,r) \in \mathrm{S}_1(APr_A)^2 : d(q,r) \leq \frac{1}{3}\}$ and $D_1 = D \cap X_{\aleph_1}^2$. Then D is closed, being the image of the closed set $[d(x,y) \leq \frac{1}{3}] \subseteq \mathrm{S}_2(APr_A)$ under the projection $\mathrm{S}_2(APr_A) \to \mathrm{S}_1(APr_A)^2$ (a continuous mapping from a compact space to a Hausdorff space is always closed). Thus D_1 is closed in $X_{\aleph_1}^2$.

Through the end of the proof we work in $X_{\aleph_1}$, with the induced topology. In particular, if $Y \subseteq X_{\aleph_1}$ then Y° denotes the interior of Y in this topology.

We start with $\pi_\varnothing(x)$ being the partial type defining $X_{\aleph_1}$. It has the property that $[\pi_\varnothing]^\circ \neq \varnothing$. Assume now we have π_s such that $[\pi_s]^\circ \neq \varnothing$. The interior has diameter $\geq \frac{1}{2}$, so there are $q, r \in [\pi_s]^\circ$ such that $d(r, q) > \frac{1}{3}$. Thus $(q, r) \notin D_1$, so they admit open neighbourhoods $q \in U \subseteq [\pi_s]^\circ$ and $r \in V \subseteq [\pi_s]^\circ$ such that $(U \times V) \cap D_1 = \varnothing$. We can then find smaller open neighbourhoods such that $q \in U_1 \subseteq \overline{U}_1 \subseteq U$ and $r \in V_1 \subseteq \overline{V}_1 \subseteq V$. Letting $\pi_{s\frown 0}$ be the partial type defining $\overline{U}_1$ and $\pi_{s\frown 1}$ the partial type defining $\overline{V}_1$ we get: $[\pi_s] \supseteq [\pi_{s\frown 0}] \supseteq [\pi_{s\frown 0}]^\circ \neq \varnothing$ and $[\pi_s] \supseteq [\pi_{s\frown 1}] \supseteq [\pi_{s\frown 1}]^\circ \neq \varnothing$. Finally, $([\pi_{s\frown 0}] \times [\pi_{s\frown 1}]) \cap D_1 = \varnothing$ implies $d(\pi_{s\frown 0}, \pi_{s\frown 1}) > \frac{1}{3}$.

Repeating this argument we obtain the required partial types. $\blacksquare_{3.4}$

Lemma 3.5 *The theory APr_A λ-stable if and only if $\lambda^{\aleph_0} = \lambda$.*

Proof One direction is since APr_A is stable in a countable language.

For the other, assume $\lambda^{\aleph_0} > \lambda$. Let $\{\pi_s : s \in 2^{<\omega}\}$ be as in Proposition 3.4. Let $\{a_i : i < \lambda\}$ be a sequence of independent events of measure $\frac{1}{2}$, all fixed by τ, and let $\mathscr{A}$ be the generated complete algebra.

For $\theta \in \lambda^{\mathbb{N}}$ and $s \in 2^{<\omega}$ let $b_{\theta,s} = \bigwedge_{i<|s|} a_{\theta(i)}^{s(i)} \in \mathscr{A}$, and let

$$\rho_{\theta,n} = \bigcup_{s \in 2^n} \pi_s^{b_{\theta,s}}(x), \qquad \rho_\theta = \bigcup_n \rho_{\theta,n}.$$

In other words, $\rho_\theta(c)$ holds if and only if, for every $s \in 2^{<\omega}$, c satisfies π_s over $\bigwedge_{i<|s|} a_{\theta(i)}^{s(i)}$. It is easy to check that $\rho_{\theta,n}$ is consistent and implies $\rho_{\theta,m}$ for $m < n$, so ρ_θ is consistent as well. Choose for each θ a complete type $r_\theta \in \mathrm{S}_1(\mathscr{A})$ extending ρ_θ.

Let $\theta \neq \theta' \in \lambda^{\mathbb{N}}$, and let $i \in \mathbb{N}$ be such that $\theta(i) \neq \theta'(i)$. Then over $a_{\theta(i)} \smallsetminus a_{\theta'(i)}$, η_{r_θ} takes only values in $\bigcup_{s \in 2^i} [\pi_{s\frown 0}] \subseteq \mathrm{S}_1(APr_A)$, while $\eta_{r_{\theta'}}$ only takes values in $\bigcup_{s \in 2^i} [\pi_{s\frown 1}]$, and the opposite holds over $a_{\theta'(i)} \smallsetminus a_{\theta(i)}$. Thus $d_\varnothing \circ (\eta_{r_\theta}, \eta_{r_{\theta'}}) \geq \frac{1}{3}$ over $a_{\theta(i)} \triangle a_{\theta'(i)}$, which has measure $\frac{1}{2}$. We conclude that $d(r_\theta, r_{\theta'}) \geq \frac{1}{6}$.

We have shown that there are $\lambda^{\mathbb{N}}$ equally distanced types over a set of λ parameters, as desired. $\blacksquare_{3.5}$

We conclude:

Theorem 3.6 *The theory APr_A is not superstable, and therefore not supersimple.*

Notice the difference from the case of IHS_A, which is superstable (but not $\aleph_0$-stable).

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